Math 291-3: MENU Linear Algebra and Multivariable Calculus
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Lecture 1: Local Extrema

Learning Objectives:

- Locate the critical points of a scalar-valued function on $\mathbb{R}^n$.
- Classify non-degenerate critical points as a local maximum, local minimum, or saddle point using the Second Derivative Test.

Welcome to MATH 291-3! Last quarter we spent a considerable amount of time and effort understanding diagonalization of square matrices. This culminated in the proof of the Spectral Theorem, which says that every symmetric real matrix is orthogonally diagonalizable. We also studied the elements of multivariable differential calculus, which was linked to the linear algebra through the interpretation of differentiability as "having a good affine approximation". The next few lectures continue to draw on the links between differentiation and linear algebra to establish the multivariable analogues of optimization (finding the maximum and minimum values of a function on a set). Once we have explored optimization of multivariable functions, we will turn our attention to multivariable integral calculus for the remainder of the course, with the ultimate goal of understanding the various types of "integration" that exist and the appropriate way(s) to understand the "fundamental theorem of calculus" in the multivariable setting.

Local Extreme Values

To simplify the notation, throughout this section we will assume that $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ is a scalar-valued function.

Definition 1. Suppose that f is defined on an open set containing $\vec{a}$. We say that f has a **local maximum value** at $\vec{a}$ if

$$f(\vec{x}) \leq f(\vec{a}) \quad \text{for every } \vec{x} \text{ in an open ball centered at } \vec{a}.$$

Similarly, f has a **local mimimum value** at $\vec{a}$ if

$$f(\vec{x}) \geq f(\vec{a}) \quad \text{for every } \vec{x} \text{ in an open ball centered at } \vec{a}.$$

We say f has a **local extreme value** at $\vec{a}$ if f either has a local maximum or local minimum value at $\vec{a}$.

Just as in single-variable calculus, the local extreme values of a differentiable function can only occur when the derivative is zero.

Theorem 1 (Fermat)**.** If f is differentiable and has a local extreme value at $\vec{a}$, then $Df(\vec{a}) = 0_{1\times n}$.

Proof. Suppose that f has a local maximum value at $\vec{a}$ and is differentiable at $\vec{a}$. (The argument when f has a local minimum value is similar.) Let $1 \leq k \leq n$. We show that $\frac{\partial f}{\partial x_k}(\vec{a}) = 0$.

To do this, note that $\frac{\partial f}{\partial x_k}(\vec{a}) = g'(0)$, where $g(t) = f(\vec{a} + t\vec{e}_k)$. The Chain Rule implies that g is differentiable at 0, and

$$g'(0) = Df(\vec{a} + 0\vec{e}_j)\vec{e}_j = Df(\vec{a})\vec{e}_j = \frac{\partial f}{\partial x_k}(\vec{a}).$$

Because f has a local maximum value at $\vec{a}$, $g(t) = f(\vec{a} + t\vec{e}_j) \leq f(\vec{a}) = g(0)$ for t near 0. In other words, the single-variable function g also has a local maximum value at 0. Because g is differentiable at 0, $g'(0) = 0$. Therefore $\frac{\partial f}{\partial x_k}(\vec{a}) = 0$ as well, and the proof is complete. □

Remark 1. Therefore, just as in single-variable calculus, the only points $\vec{a}$ at which f can have a local extreme value satisfy either

(i) f is differentiable at $\vec{a}$ and $Df(\vec{a}) = 0_{1\times n}$

(ii) f is not differentiable at $\vec{a}$.

Such a point $\vec{a}$ is called a **critical point** of f. In practice most of the functions we deal with will be differentiable (i.e. all critical points will fall under case (i)), but case (ii) can happen as well.

Example 1. The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \sqrt{x^2 + y^2}$ is C^1 on $\mathbb{R}^2 - \{(0, 0)\}$, and is therefore differentiable everywhere except possibly at $(0, 0)$. You can check that f is not differentiable at $(0, 0)$ (since, for example, $f_x(0, 0)$ does not exist), so that $(0, 0)$ is a critical point of f. Moreover, for $(x, y) \neq (0, 0)$, $Df(x, y) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix}$, so (x, y) is not a critical point of f. Therefore the only point at which f might have a local extreme value is $(0, 0)$.

Indeed, f has a local minimum value at $(0, 0)$ because $f(0, 0) = 0 \leq \sqrt{x^2 + y^2} = f(x, y)$ for every $(x, y) \in \mathbb{R}^2$.

Example 2. Identify the critical points of the function $f(x, y) = y^2 - x^2 - x^3 - x^2y$.

Here we note that f is C^1 (and therefore differentiable) on $\mathbb{R}^2$, so the critical point(s) (x, y) of f must satisfy

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = Df(x, y) = \begin{bmatrix} -2x - 3x^2 - 2xy & 2y - x^2 \end{bmatrix}, \text{ or rather } 0 = -x(2 + 3x + 2y) \text{ and } 0 = 2y - x^2.$$

The first equation implies that $x = 0$ or $2 + 3x + 2y = 0$. In light of the second equation, if $x = 0$ then $y = \frac{1}{2}x^2 = 0$, and one can verify that $(0, 0)$ is indeed a critical point of f.

On the other hand, if $2 + 3x + 2y = 0$ then we can substitute $y = \frac{1}{2}x^2$ (from the second equation) to see that $0 = x^2 + 3x + 2 = (x + 2)(x + 1)$, so that $x = -2$ or $x = -1$. If $x = -2$ then $y = \frac{1}{2}(-2)^2 = 2$. If $x = -1$ then $y = \frac{1}{2}(-1)^2 = \frac{1}{2}$. One can indeed verify that $(-2, 2)$ and $(-1, \frac{1}{2})$ are critical points of f.

To summarize, the critical points of f are exactly $(0, 0)$, $(-2, 2)$, and $(-1, \frac{1}{2})$.

Example 3. It is not the case that $Df(\vec{a}) = 0_{1\times n}$ necessarily implies that f has a local extreme value at $\vec{a}$. Indeed, it is possible for every ball centered at $\vec{a}$ to contain points $\vec{x}$ and $\vec{y}$ with $f(\vec{y}) > f(\vec{a})$ and $f(\vec{x}) < f(\vec{a})$. Such a critical point $\vec{a}$ is called a **saddle point** of f.

For an example where this occurs, consider the function $f(x, y) = x^2 - y^2$. Then f is C^1 (and therefore differentiable) with $Df(0, 0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$, so that $(0, 0)$ is a critical point of f. On the other hand, $f(x, 0) = x^2 > 0 = f(0, 0)$ for every $x \neq 0$ and $f(0, y) = -y^2 < 0 = f(0, 0)$ for every $y \neq 0$. Therefore f has neither a local maximum value nor a local minimum value at $(0, 0)$, so that $(0, 0)$ is a saddle point of f.

Characterizing A Critical Point

In single-variable calculus one can use the second derivative of a C^2 function to characterize the behavior of the function at a critical point: if $f'(a) = 0$ and $f''(a) > 0$ then f has a local minimum value at a (if $f''(a) < 0$, then f has a local maximum value at a). In your single-variable calculus course this was called the "Second Derivative Test", and was likely explained by appealing to concavity. In the multivariable setting we instead appeal to our approximation

$$f(\vec{x}) \approx f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a}) \cdot (D^2f(\vec{a})(\vec{x}-\vec{a}))$$

for $\vec{x}$ near $\vec{a}$. If $\vec{a}$ is a critical point of f, then $Df(\vec{a}) = 0_{1\times n}$, so for $\vec{x}$ near $\vec{a}$ we have

$$f(\vec{x}) \approx f(\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a}) \cdot (D^2f(\vec{a})(\vec{x}-\vec{a})).$$

This suggests that the quadratic form on the right (with matrix $D^2f(\vec{a})$) should capture the qualitative behavior of $f(\vec{x}) - f(\vec{a})$ for $\vec{x}$ near $\vec{a}$.

To make this precise, we will need to know that f is C^2 in an open ball centered at $\vec{a}$, and that the Hessian of f at $\vec{a}$ is invertible. With this in mind, we make the following definition.

Definition 2. Suppose that $\vec{a}$ is a critical point of f and that f is C^2 in an open ball centered at $\vec{a}$. If $D^2f(\vec{a})$ is invertible, then we say that $\vec{a}$ is a **non-degenerate** critical point of f. If $D^2f(\vec{a})$ is not invertible, then we say that $\vec{a}$ is a **degenerate** critical point of f.

Recall that a real symmetric matrix A (and also its associated quadratic form $\vec{x} \mapsto \vec{x}\cdot(A\vec{x})$) is called **positive definite** if all of the eigenvalues of A are positive, **negative definite** if all of the eigenvalues of A are negative, and **indefinite** if A has at least one positive eigenvalue and at least one negative eigenvalue.

The multidimensional Second Derivative Test can therefore be stated as follows.

Theorem 2 (Second Derivative Test)**.** Let $\Omega \subseteq \mathbb{R}^n$, let $\vec{a} \in \Omega$, assume $f : \Omega \to \mathbb{R}$ is C^2 in an open ball centered at $\vec{a}$, and assume that $\vec{a}$ is a non-degenerate critical point of f.

(i) If $D^2f(\vec{a})$ is positive definite, then f has a local minimum value at $\vec{a}$.

(ii) If $D^2f(\vec{a})$ is negative definite, then f has a local maximum value at $\vec{a}$.

(iii) If $D^2f(\vec{a})$ is indefinite, then f has a saddle point at $\vec{a}$.

Proof. We will prove the result under the assumption that f is C^3. (The proof under the weaker assumption that f is C^2 is more technical, but not much more enlightening.)

Suppose that f is C^3 on some ball $B_\delta(\vec{a})$. Note that $B \stackrel{def}{=} \{\vec{y} \;:\; \|\vec{y}-\vec{a}\| \leq \frac{\delta}{2}\}$ is a compact subset of $B_\delta(\vec{a})$. Because f is C^3 and B is compact, there is a constant $M > 0$ such that for each third-order partial derivative $f_{x_ix_jx_k}$ of f, $|f_{x_ix_jx_k}(\vec{y})| \leq M$ for every $\vec{x} \in B$. (We'll use this estimate later on in the proof.)

Fix $\vec{x} \in B$, define $g(t) = f(\vec{a} + t(\vec{x}-\vec{a}))$. By the one-variable Taylor Formula, there is $c \in [0,1]$ such that

$$g(1) = g(0) + g'(0)1 + \frac{1}{2!}g''(0)1^2 + \frac{1}{3!}g'''(c)1^3 = g(0) + g'(0) + \frac{1}{2}g''(0) + \frac{1}{6}g'''(c).$$

The equation above is secretely the same as

$$f(\vec{x}) = f(\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a}) \cdot (D^2 f(\vec{a})(\vec{x}-\vec{a})) + \frac{1}{6}g'''(c).$$

To see why, note that $g(1) = f(\vec{x})$, $g(0) = f(\vec{a})$, and (by the chain rule) we have

$$g'(t) = Df(\vec{a}+t(\vec{x}-\vec{a}))(\vec{x}-\vec{a}) = \sum_{i=1}^{n} f_{x_i}(\vec{a}+t(\vec{x}-\vec{a}))(x_i - a_i)$$

and

$$g''(t) = \sum_{i=1}^{n} \Big(Df_{x_i}(\vec{a}+t(\vec{x}-\vec{a}))(\vec{x}-\vec{a})\Big)(x_i-a_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{x_i x_j}(\vec{a}+t(\vec{x}-\vec{a}))(x_j-a_j)(x_i-a_i)$$

and

$$\begin{aligned} g'''(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(Df_{x_i x_j}(\vec{a}+t(\vec{x}-\vec{a}))(\vec{x}-\vec{a})\Big)(x_j-a_j)(x_i-a_i) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f_{x_i x_j x_k}(\vec{a}+t(\vec{x}-\vec{a}))(x_k-a_k)(x_j-a_j)(x_i-a_i). \end{aligned}$$

Then because $\vec{a}$ is a critical point of f,

$$g'(0) = Df(\vec{a})(\vec{x}-\vec{a}) = 0_{1\times n}(\vec{x}-\vec{a}) = 0.$$

Also

$$g''(0) = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{x_i x_j}(\vec{a})(x_j-a_j)(x_i-a_i) = (\vec{x}-\vec{a}) \cdot \begin{bmatrix} \sum_{j=1}^{n} f_{x_1 x_j}(\vec{a})(x_j-a_j) \\ \sum_{j=1}^{n} f_{x_2 x_j}(\vec{a})(x_j-a_j) \\ \vdots \\ \sum_{j=1}^{n} f_{x_n x_j}(\vec{a})(x_j-a_j) \end{bmatrix} = (\vec{x}-\vec{a}) \cdot (D^2 f(\vec{a})(\vec{x}-\vec{a})).$$

We have therefore shown that

$$f(\vec{x}) = f(\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a}) \cdot (D^2 f(\vec{a})(\vec{x}-\vec{a})) + \frac{1}{6}g'''(c).$$

We think of $\frac{1}{6}g'''(c)$ as an "error" term that represents the difference between $f(\vec{x})$ and the polynomial $f(\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a}) \cdot (D^2 f(\vec{a})(\vec{x}-\vec{a}))$. We would like to say that the behavior of $f(\vec{x})$ is the same as the behavior of $f(\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a}) \cdot (D^2 f(\vec{a})(\vec{x}-\vec{a}))$ for $\vec{x}$ near $\vec{a}$, but to do this we need to know that $\frac{1}{6}g'''(c)$ is smaller than $\frac{1}{2}(\vec{x}-\vec{a}) \cdot (D^2 f(\vec{a})(\vec{x}-\vec{a}))$.

We first find an upper bound for the size of $g'''(c)$. Since $\vec{a}+c(\vec{x}-\vec{a}) \in B$ and $|x_m - a_m| \le \|\vec{x}-\vec{a}\|$ for each $1 \le m \le n$,

$$\begin{aligned} |g'''(c)| &\le \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |f_{x_i x_j x_k}(\vec{a}+c(\vec{x}-\vec{a}))||x_k-a_k||x_j-a_j||x_i-a_i| \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} M\|\vec{x}-\vec{a}\|\|\vec{x}-\vec{a}\|\|\vec{x}-\vec{a}\| \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} M\|\vec{x}-\vec{a}\|^3 = \sum_{i=1}^{n}\sum_{j=1}^{n} nM\|\vec{x}-\vec{a}\|^3 = \sum_{i=1}^{n} n^2 M\|\vec{x}-\vec{a}\|^3 = n^3 M\|\vec{x}-\vec{a}\|^3, \end{aligned}$$

so that $-n^3M\|\vec{x}-\vec{a}\|^3 \le g'''(c) \le n^3M\|\vec{x}-\vec{a}\|^3$.

Next we analyze $(\vec{x}-\vec{a})\cdot(D^2f(\vec{a})(\vec{x}-\vec{a}))$ (the proof that this is "not too small" will come later). Recall that because f is C^2, $D^2f(\vec{a})$ is symmetric. By the Spectral Theorem, $D^2f(\vec{a})$ has an orthonormal eigenbasis $\vec{u}_1,\ldots,\vec{u}_n$. For each $1\le k\le n$, let λ_k be the eigenvalue of $D^2f(\vec{a})$ associated to $\vec{u}_k$. Then

$$(\vec{x}-\vec{a})\cdot(D^2f(\vec{a})(\vec{x}-\vec{a})) = \lambda_1((\vec{x}-\vec{a})\cdot\vec{u}_1)^2+\cdots+\lambda_n((\vec{x}-\vec{a})\cdot\vec{u}_n)^2.$$

We are now in a position to prove the various statements in the theorem. Suppose that $D^2f(\vec{a})$ is positive definite. Then if $\lambda>0$ denotes the smallest eigenvalue of $D^2f(\vec{a})$, we have

$$\begin{aligned}(\vec{x}-\vec{a})\cdot(D^2f(\vec{a})(\vec{x}-\vec{a})) &= \lambda_1((\vec{x}-\vec{a})\cdot\vec{u}_1)^2+\cdots+\lambda_n((\vec{x}-\vec{a})\cdot\vec{u}_n)^2\\ &\ge \lambda((\vec{x}-\vec{a})\cdot\vec{u}_1)^2+\cdots+\lambda((\vec{x}-\vec{a})\cdot\vec{u}_n)^2\\ &= \lambda(((\vec{x}-\vec{a})\cdot\vec{u}_1)^2+\cdots+((\vec{x}-\vec{a})\cdot\vec{u}_n)^2)\\ &= \lambda\|\vec{x}-\vec{a}\|^2.\end{aligned}$$

Therefore for $\vec{x}\in B$ we have

$$\begin{aligned}f(\vec{x}) &= f(\vec{a})+\frac{1}{2}(\vec{x}-\vec{a})\cdot(D^2f(\vec{a})(\vec{x}-\vec{a}))+\frac{1}{6}g'''(c)\\ &\ge f(\vec{x})+\frac{1}{2}\lambda\|\vec{x}-\vec{a}\|^2-\frac{1}{6}n^3M\|\vec{x}-\vec{a}\|^3\\ &= f(\vec{a})+\frac{1}{2}\|\vec{x}-\vec{a}\|^2\Big(\lambda-\frac{n^3M}{3}\|\vec{x}-\vec{a}\|\Big).\end{aligned}$$

Therefore, as long as

$$\lambda-\frac{n^3M}{3}\|\vec{x}-\vec{a}\|\ge 0,\quad \text{or rather } \|\vec{x}-\vec{a}\|\le\frac{3\lambda}{n^3M},$$

then we have $f(\vec{x})\ge f(\vec{a})$. Since we already needed $\|\vec{x}-\vec{a}\|\le\frac{\delta}{2}$ at the beginning of the proof, we see that $f(\vec{x})\ge f(\vec{a})$ as long as $\|\vec{x}-\vec{a}\|\le\min\left(\frac{\delta}{2},\frac{3\lambda}{n^3M}\right)$. Therefore f has a local minimum value at $\vec{a}$.

The proof that f has a local maximum value at $\vec{a}$ is $D^2f(\vec{a})$ is negative definite follows from a similar argument, but with replacing λ with the largest eigenvalue of $D^2f(\vec{a})$ (note that $\lambda<0$ if $D^2f(\vec{a})$ is negative definite).

If $D^2f(\vec{a})$ is indefinite, then by relabelling the vectors $\vec{u}_1,\ldots,\vec{u}_n$ we can arrange it so that $\lambda_1>0$ and $\lambda_2<0$. Then

$$f(\vec{a}+t\vec{u}_1)\ge f(\vec{a})+\frac{t^2\lambda_1}{2}-\frac{1}{6}n^3Mt^3 = f(\vec{a})+\frac{t^2}{2}\Big(\lambda_1-\frac{n^3Mt}{3}\Big)>f(\vec{a})$$

as long at $0<t<\min(\frac{\delta}{2},\frac{3\lambda_1}{n^3M})$. Therefore every ball centered at $\vec{a}$ contains a point $\vec{x}$ for which $f(\vec{x})>f(\vec{a})$. By replacing $\vec{u}_1$ with $\vec{u}_2$, we see that

$$f(\vec{a}+t\vec{u}_2)\le f(\vec{a})+\frac{t^2\lambda_2}{2}+\frac{1}{6}n^3Mt^3 = f(\vec{a})+\frac{t^2}{2}\Big(\lambda_2+\frac{n^3M}{3}t\Big)<f(\vec{a})$$

as long as $0<t<\frac{-3\lambda_2}{n^3M}$ (remember that $\lambda_2<0$!). Therefore every ball centered at $\vec{a}$ contains a point $\vec{x}$ for which $f(\vec{x})<f(\vec{a})$, and f has a saddle point at $\vec{a}$. $\square$

Example 4. Recall that the function $f(x,y) = y^2 - x^2 - x^3 - x^2y$ has critical points at $(0,0)$, $(-2,2)$, and $(-1, \frac{1}{2})$. Let's use the Second Derivative Test to classify each of these critical points.

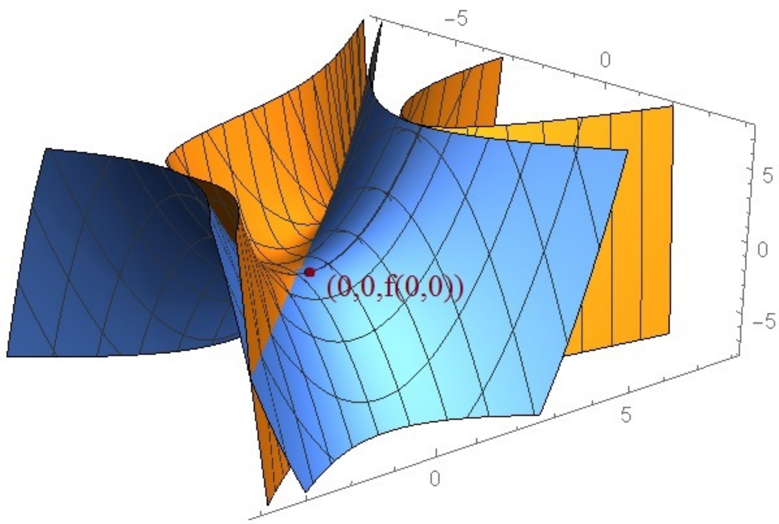
We first compute that

$$D^2f(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix} = \begin{bmatrix} -2-6x-2y & -2x \\ -2x & 2 \end{bmatrix}.$$

At the critical point $(0,0)$ we have $D^2f(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$, which has eigenvalues -2 and 2. Therefore $D^2f(0,0)$ is invertible (so that the critical point $(0,0)$ is non-degenerate), and $D^2f(0,0)$ is indefinite. Therefore the Second Derivative Test implies that f has a saddle point at $(0,0)$. To picture what is going on here, note that the conditions of the Second Derivative Test tell us that

$$f(0,0) + \frac{1}{2}(\vec{x} - \vec{0}) \cdot (D^2f(0,0)(\vec{x} - \vec{0})) = 0 + \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} \cdot \left(\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) = -x^2 + y^2$$

should be a good approximation for f near $(0,0)$, and that f should have the same type of critical point at $(0,0)$ as does the quadratic polynomial $-x^2+y^2$. The graphs of $z = f(x,y)$ (orange) and $z = -x^2+y^2$ (blue) are superimposed in the picture below.

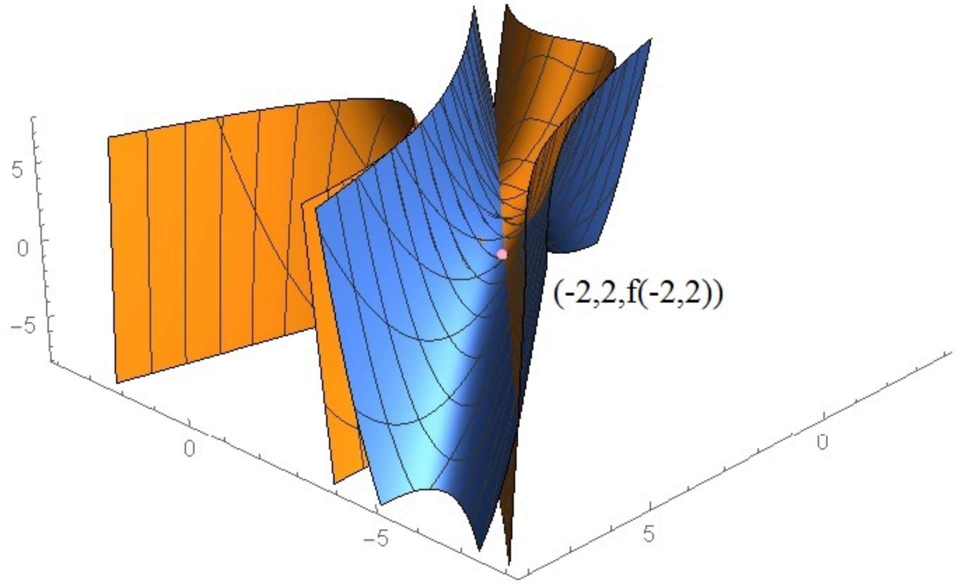


At the critical point $(-2,2)$ we have $D^2f(-2,2) = \begin{bmatrix} 6 & 4 \\ 4 & 2 \end{bmatrix}$. Because this matrix is diagonalizable its characteristic polynomial factors as $\lambda^2 - 8\lambda - 4 = (\lambda_1 - \lambda)(\lambda_2 - \lambda)$, where λ_1, λ_2 are the (not necessarily distinct) eigenvalues of $D^2f(-2,2)$. Evaluating[1] at 0 gives $-4 = \lambda_1\lambda_2$, so that one of λ_1, λ_2 is positive and the other is negative. Therefore $D^2f(-2,2)$ is indefinite, so that f has a saddle point at $(-2,2)$. Again, we plot the graphs of $z = f(x,y)$ (orange) and the hyperbolic paraboloid

$$z = f(-2,2) + \frac{1}{2}\begin{bmatrix} x+2 \\ y-2 \end{bmatrix} \cdot \left(\begin{bmatrix} 6 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x+2 \\ y-2 \end{bmatrix} \right) = 3(x+2)^2 + 4(x+2)(y-2) + (y-2)^2$$

(blue) near $(-2,2)$ in the picture below.

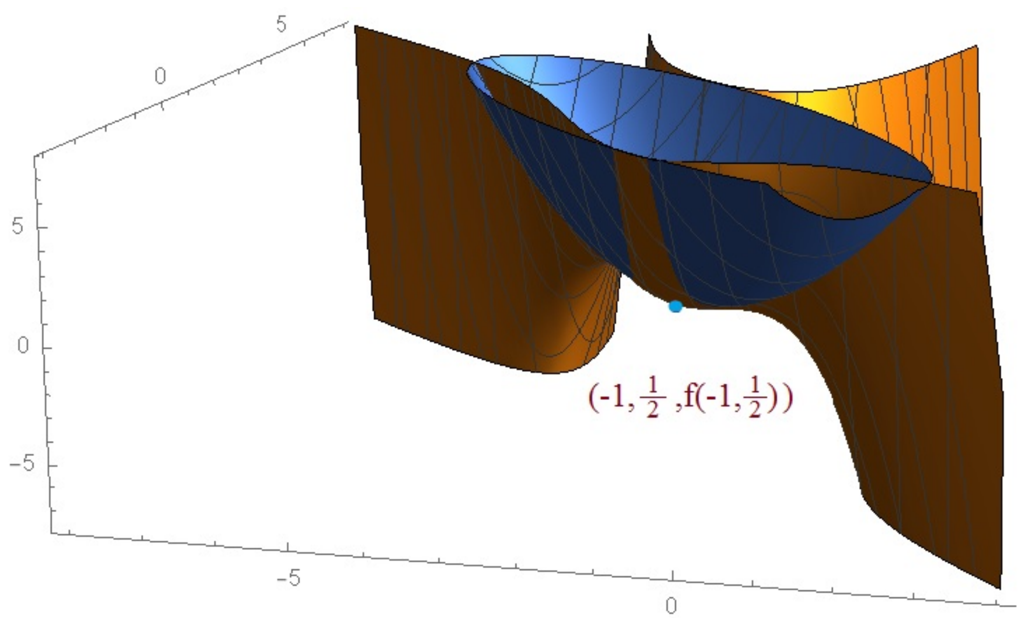
[1]Note that we are essentially proving an altered version of a result that we had last quarter: if $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\det(A) = \lambda_1^{\text{almu}(\lambda_1)} \cdots \lambda_k^{\text{almu}(\lambda_k)}$. Last quarter we proved this result for complex matrices without the assumption of diagonalizability, but the same proof works for this case as well. The only thing we need to know is that the characteristic polynomial of A factors completely into first-order factors, and that holds for $n \times n$ diagonalizable matrices because the sum of the geometric multiplicities of the eigenvalues (and therefore the sum of the algebraic multiplicities) is equal to n.



At the critical point $(-1, \frac{1}{2})$ we have $D^2 f(-1, \frac{1}{2}) = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ which has characteristic equation $\lambda^2 - 5\lambda + 2 = 0$. The roots of this equation are $\lambda = \frac{5 \pm \sqrt{25-8}}{2}$. Because $0 < \sqrt{25-8} < \sqrt{25} = 5$, $\frac{5+\sqrt{25-8}}{2} > 0$ and $\frac{5-\sqrt{25-8}}{2} > 0$, so that $D^2 f(-1, \frac{1}{2})$ is positive definite. Therefore the Second Derivative Test implies that f has a local minimum at $(-1, \frac{1}{2})$. Here is a plot of the graph of $z = f(x, y)$ (orange) and the elliptic paraboloid

$$z = f\Big(-1, \frac{1}{2}\Big) + \frac{1}{2} \begin{bmatrix} x+1 \\ y-\frac{1}{2} \end{bmatrix} \cdot \Big(\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x+1 \\ y-\frac{1}{2} \end{bmatrix} \Big) = -\frac{1}{4} + \frac{3}{2}(x+1)^2 + 2(x+1)\Big(y - \frac{1}{2}\Big) + \Big(y - \frac{1}{2}\Big)^2$$

(blue) near $(-1, \frac{1}{2})$:



Lecture 2: Global Extrema

Learning Objectives:

- Determine the global extreme values of a continuous function on a compact set.

Although local extreme values are particularly nice to study because they are detectable (and sometimes classifiable) using differential calculus, in practice one is usually concerned with the **global** (or **absolute**) extreme values of a function on a set.

Definition 3. Let $E \subseteq \Omega \subseteq \mathbb{R}^n$, let $f : \Omega \to \mathbb{R}$, and let $\vec{a} \in E$. We say that f has a **global minimum value** on E at $\vec{a}$ if $f(\vec{x}) \geq f(\vec{a})$ for every $\vec{x} \in E$. Similarly, we say that f has a **global maximum value** on E at $\vec{a}$ if $f(\vec{x}) \leq f(\vec{a})$ for every $\vec{x} \in E$.

The global minimum and maximum values of f on E (when they exist) are unique. They are *the* smallest and largest values of f on E. It is not always true that every function attains global extreme values on a set, and it is also not the case that the concepts of "global extreme value" and "local extreme value" interact as nicely as you'd expect.

Example 5. Let $f(x,y) = x^2(1-y)^3 + y^2$. Then $(0,0)$ is the unique critical point of f on $\mathbb{R}^2$, and f has a local minimum value at $(0,0)$, but f does not have a global minimum value on $\mathbb{R}^2$.

To see this, note that f is C^1 (indeed, C^k for every k) on $\mathbb{R}^2$ and therefore differentiable, and that

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = Df(x,y) = \begin{bmatrix} 2x(1-y)^3 & -3x^2(1-y)^2 + 2y \end{bmatrix}$$

exactly when $0 = 2x(1-y)^3$ and $0 = -3x^2(1-y)^2 + 2y$. The first equation requires that either $x = 0$ or $y = 1$. If $x = 0$ then the second equation simplifies to $0 = 2y$, so we must have $y = 0$ as well. Therefore $(0,0)$ is a critical point of f. If $x \neq 0$ then $y = 1$, but then the second equation simplifies to $0 = -3x^2(0)^2 + 2 = 2$, an impossibility. Therefore $(0,0)$ is the only critical point of f.

Note also that

$$D^2f(x,y) = \begin{bmatrix} 2(1-y)^3 & -6x(1-y)^2 \\ -6x(1-y)^2 & 6x^2(1-y) + 2 \end{bmatrix}, \quad \text{so that} \quad D^2f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore all eigenvalues of $D^2f(0,0)$ are positive, and the Second Derivative Test implies that f has a local minimum value at $(0,0)$.

But $f(0,0) = 0$ is not the global minimum value of f on $\mathbb{R}^2$, since (for example) $f(1,4) = (1-4)^3 + 4^2 = -27 + 16 = -11 < 0 = f(0,0)$. (Indeed, f has no global minimum value on $\mathbb{R}^2$, since $f(1,y) = (1-y)^3 + y^2$ approaches $-\infty$ as $y \to \infty$.)

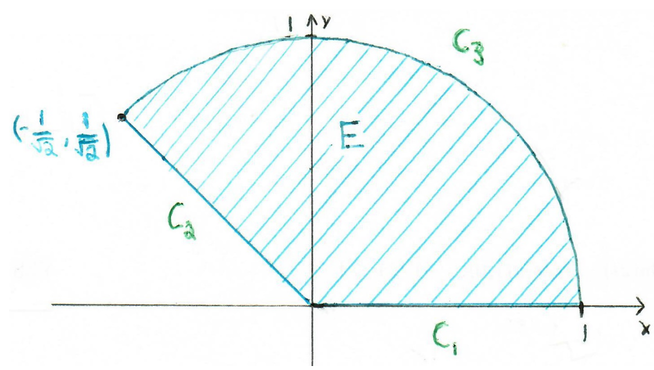
Finding Global Extrema

There is one circumstance under which we can be certain that a function f has global extreme values on a set $E \subseteq \mathbb{R}^n$: when f is continuous on E and E is compact. This is a direct consequence of the Extreme Value Theorem.

For such a function f, if the global maximum (or minimum) on E occurs at a point $\vec{a} \in E$ that is not on the boundary of E, then f will have a $f(\vec{a})$ will be a local extreme value of f and therefore $\vec{a}$ will be a critical point of f.

If the global maximum (or minimum) of f on E occurs at a point on the boundary of E, then that point need not be a critical point of f and we will need some other way to detect it besides checking Df. One technique for this is to parameterize pieces of the boundary (we'll see how this is done in the next example), and another is to use a more sophisticated technique known as the Method of Lagrange Multipliers (discussed next time). In either case, we will note the following (very important) point: if the global maximum (or minimum) of f on E occurs on ∂E, then this value will also be the global maximum (or minimum) of f when viewed as a function on ∂E.

Example 6. Find the global minimum and maximum values of $f(x, y) = 2x^2 + 2y^2 - xy - 3y - 3x + 2$ on the region E pictured below (the curved portion of the boundary of E is part of the unit circle):



Note that f is C^1 (and therefore differentiable) throughout $\mathbb{R}^2$, and that

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = Df(x, y) = \begin{bmatrix} 4x - y - 3 & 4y - x - 3 \end{bmatrix}$$

exactly when (x, y) satisfies $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, or when $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so at $(1, 1)$. Because $(1, 1)$ does not lie in E, we conclude that f does not have any local extreme values on E.

We turn our attention to the boundary of E, which is comprised of three curves: C_1 (the portion of the positive x-axis between the origin and $(1, 0)$), C_2 (the portion of the line $y = -x$ in the second quadrant between $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and the origin), and C_3 (the portion of the circle $x^2 + y^2 = 1$ in the upper-half plane between the positive x-axis and the line $y = -x$). On each piece C_1, C_2, and C_3, we write f as a one-variable function and determine where it may have its maximum or minimum values.

On C_1 we can think of f as a function of one variable by writing $h(x) = f(x, 0) = 2x^2 - 3x + 2$ for $0 \leq x \leq 1$. The critical numbers of h are solutions to $0 = h'(x) = 4x - 3$, or rather $x = \frac{3}{4}$. Since $0 \leq \frac{3}{4} \leq 1$, we will test h at $x = \frac{3}{4}$ in addition to the endpoints $x = 0$ and $x = 1$ of the interval $[0, 1]$. Therefore, the only possible points at which h might have an global maximum or minimum value are $x = 0$, $x = \frac{3}{4}$, and $x = 1$. Therefore the only points on C_1 at which f could have global extreme values are $(0, 0)$, $(\frac{3}{4}, 0)$, and $(1, 0)$.

On C_2 we have $y = -x$, and therefore our simplified version of f is $h(x) = f(x, -x) = 5x^2 + 2$ for $-\frac{1}{\sqrt{2}} \leq x \leq 0$. The critical numbers of h are solutions to $0 = h'(x) = 10x$, or rather $x = 0$. Since 0 is in the interval $[-\frac{1}{\sqrt{2}}, 0]$, the possible points at which h might have an global maximum or minimum value are the endpoints $x = 0$ and $x = -\frac{1}{\sqrt{2}}$. Therefore the only points on C_2 at which f could have global extreme values are $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(0, 0)$.

On C_3 we can represent f as a function of one variable by first noting that the path $\vec{r}(t) = (\cos(t), \sin(t))$, $0 \le t \le \frac{3\pi}{4}$ traces out C_3, and therefore we want to determine where

$$\begin{aligned} h(t) = f(\vec{r}(t)) &= 2\cos^2(t) + 2\sin^2(t) - \cos(t)\sin(t) - 3\sin(t) - 3\cos(t) + 2 \\ &= -\cos(t)\sin(t) - 3\sin(t) - 3\cos(t) + 4, \qquad 0 \le t \le \frac{3\pi}{4} \end{aligned}$$

might achieve its maximum or minimum values. The critical numbers of h all satisfy

$$\begin{aligned} 0 &= h'(t) \\ &= \sin^2(t) - \cos^2(t) - 3\cos(t) + 3\sin(t) \\ &= (\sin(t) - \cos(t))(\sin(t) + \cos(t)) + 3(\sin(t) - \cos(t)) \\ &= (\sin(t) - \cos(t))(\sin(t) + \cos(t) + 3). \end{aligned}$$

Since $-2 \le \sin(t)+\cos(t)$, we have $1 \le \sin(t)+\cos(t)+3$, and therefore $\sin(t)+\cos(t)+3 \neq 0$. Therefore we must have $\sin(t) - \cos(t) = 0$, or rather $\sin(t) = \cos(t)$. The only value of t in $[0, \frac{3\pi}{4}]$ at which this occurs is $t = \frac{\pi}{4}$, and therefore we will test h at $t = \frac{\pi}{4}$ and also at the endpoints $t = 0$ and $t = \frac{3\pi}{4}$. Therefore the only points on C_3 at which f might have global extreme values are $(0, 0)$ (when $t = 0$), $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ (when $t = \frac{\pi}{4}$), and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ (when $t = \frac{3\pi}{4}$).

In total, we see that the only possible points where f could achieve its global extreme values on E must be $(0, 0)$, $(\frac{3}{4}, 0)$, $(1, 0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

We have

$$f(0,0) = 2, \qquad f\Big(\frac{3}{4}, 0\Big) = 2\Big(\frac{3}{4}\Big)^2 - 3\Big(\frac{3}{4}\Big) + 2 = \frac{7}{8}, \qquad f(1,0) = 1,$$

$$f\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big) = 1 + 1 - \frac{1}{2} - \frac{6}{\sqrt{2}} + 2 = \frac{7 - 6\sqrt{2}}{2}, \qquad f\Big(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big) = 1 + 1 + \frac{1}{2} + 2 = \frac{9}{2}.$$

Note that since $49 < 72$, taking square roots gives $7 < 6\sqrt{2}$, so that $\frac{7-6\sqrt{2}}{2} < 0$.

Therefore the global maximum value of f on E is $\frac{9}{2}$ (and occurs at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$). The global minimum value of f on E is $\frac{7-6\sqrt{2}}{2}$ (and occurs at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$).

Summary

To summarize: the global maximum and minimum values of a continuous function $f : \Omega \to \mathbb{R}$ on a compact subset $E \subseteq \Omega$ must occur at a critical point of f in E, or at a point on the boundary of E. We therefore have an algorithm for determining these points:

(i) Determine all critical points of f that lie in E.

(ii) Determine all points on ∂E at which f might have a maximum or minimum value (relative to its other values on ∂E).

(iii) Compute the value of f at each point you identified in (i) and (ii). The global maximum and minimum values of f will be among these values.

Example 7. Find the extreme values of $f(x, y) = 3x^2 + y^3$ on the region $x^2 + 2y^2 \le 1$ (this is the region bounded by the ellipse $x^2 + 2y^2 = 1$).

We first compute the critical points of f:

$$0 = f_x(x, y) = 6x \qquad \text{and} \qquad 0 = f_y(x, y) = 3y^2,$$

so the only critical point of f is $(0, 0)$ (which is within our region).

Now we determine at which points on the boundary we might have a maximum or a minimum value.

The boundary is described by the equation $x^2 + 2y^2 = 1$, so that $x^2 = 1 - 2y^2$. Substituting this into f, we obtain

$$f(y) = 3(1 - 2y^2) + y^3 = y^3 - 6y^2 + 3.$$

We also need to determine the domain of this simplified version of f. Because $2y^2 \leq 1$, the domain is $-\frac{1}{\sqrt{2}} \leq y \leq \frac{1}{\sqrt{2}}$.

Now, $f'(y) = 3y^2 - 12y = 3y(y - 4) = 0$ exactly when $y = 0$ or $y = 4$. Because $y = 4$ is not in our domain, we ignore it. This tells us that we should test the points on the ellipse corresponding to $y = 0$ (the critical point), and $y = \frac{1}{\sqrt{2}}$ and $y = -\frac{1}{\sqrt{2}}$ (the endpoints).

The points on the ellipse $x^2 + 2y^2 = 1$ which satisfy $y = 0$ are $(1, 0)$ and $(-1, 0)$. The only point satisfying $y = \frac{1}{\sqrt{2}}$ is $(0, \frac{1}{\sqrt{2}})$, and the only point satisfying $y = -\frac{1}{\sqrt{2}}$ is $(0, -\frac{1}{\sqrt{2}})$.

Therefore, in addition to $(0, 0)$, we need to test $(1, 0)$, $(-1, 0)$, $(0, \frac{1}{\sqrt{2}})$, and $(0, -\frac{1}{\sqrt{2}})$. Testing yields

$$f(0, 0) = 0, \quad f(1, 0) = 3, \quad f(-1, 0) = 3, \quad f\Big(0, \frac{1}{\sqrt{2}}\Big) = \frac{1}{2\sqrt{2}}, \quad f\Big(0, -\frac{1}{\sqrt{2}}\Big) = -\frac{1}{2\sqrt{2}}.$$

Therefore, the global maximum value of f is 3 (at $(1, 0)$ and $(-1, 0)$), while the global minimum value of f is $-\frac{1}{2\sqrt{2}}$ (at $(0, -\frac{1}{2\sqrt{2}})$).

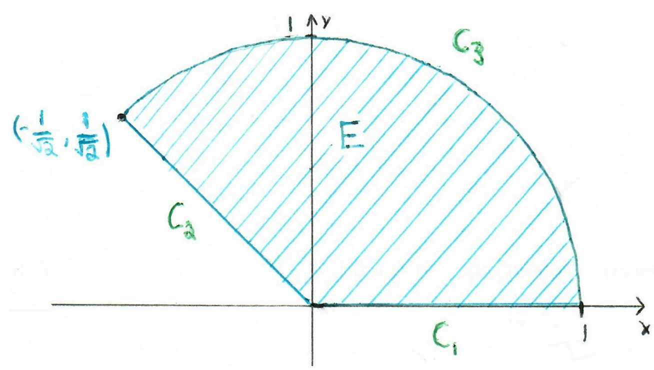
Lecture 3: Constrained Extrema

Learning Objectives:

- Explain the geometric idea behind the method of Lagrange multipliers.
- Compute the constrained extrema of a function using the method of Lagrange multipliers.

We start today by finishing up a problem from last time.

Example 8. Find the global minimum and maximum values of $f(x, y) = 2x^2 + 2y^2 - xy - 3y - 3x + 2$ on the region E pictured below (the curved portion of the boundary of E is part of the unit circle):



Last time we determined that $(1, 1)$ is the only critical point of f, but that we are uninterested in $(1, 1)$ because it lies outside of E.

We next turned our attention to finding points on ∂E at which f might have a maximum or minimum value.

On the curve C_1, we noted that f can be thought of as a one-variable function $h(x) = f(x, 0) = 2x^2 - 3x + 2$, for $x \in [0, 1]$. By applying single-variable calculus techniques, we found that h might have a maximum or minimum value when $x = 0$, $x = \frac{3}{4}$, or $x = 1$, so that if we **constrain** (i.e. restrict) the inputs of f to C_1, f might have a maximum or minimum value at $(0, 0)$, $(\frac{3}{4}, 0)$, or $(1, 0)$.

On the curve C_2, we can think of f as a one-variable function $h(x) = f(x, -x) = 5x^2 + 2$ for $x \in [-\frac{1}{\sqrt{2}}, 0]$. Again by applying single-variable calculus techniques, we found that h might have a maximum or minimum value when $x = -\frac{1}{\sqrt{2}}$ or $x = 0$. This means that if we constrain f to C_2, then f might have a maximum or minimum value at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(0, 0)$.

We now constrain the inputs of f to $(x, y) \in C_3$. Here we can represent f as a function of one variable by first noting that the path $\vec{r}(t) = (\cos(t), \sin(t))$, $0 \leq t \leq \frac{3\pi}{4}$ traces out C_3, and therefore we want to determine where

$$\begin{aligned} h(t) = f(\vec{r}(t)) &= 2\cos^2(t) + 2\sin^2(t) - \cos(t)\sin(t) - 3\sin(t) - 3\cos(t) + 2 \\ &= -\cos(t)\sin(t) - 3\sin(t) - 3\cos(t) + 4, \qquad 0 \leq t \leq \frac{3\pi}{4} \end{aligned}$$

might achieve its maximum or minimum values. The critical numbers of h all satisfy

$$\begin{aligned} 0 &= h'(t) \\ &= \sin^2(t) - \cos^2(t) - 3\cos(t) + 3\sin(t) \\ &= (\sin(t) - \cos(t))(\sin(t) + \cos(t)) + 3(\sin(t) - \cos(t)) \\ &= (\sin(t) - \cos(t))(\sin(t) + \cos(t) + 3). \end{aligned}$$

Since $-2 \leq \sin(t)+\cos(t)$, we have $1 \leq \sin(t)+\cos(t)+3$, and therefore $\sin(t)+\cos(t)+3 \neq 0$. Therefore we must have $\sin(t) - \cos(t) = 0$, or rather $\sin(t) = \cos(t)$. The only value of t in $[0, \frac{3\pi}{4}]$ at which this occurs is $t = \frac{\pi}{4}$, and therefore we will test h at $t = \frac{\pi}{4}$ and also at the endpoints $t = 0$ and $t = \frac{3\pi}{4}$. In terms of f, this means that we should test f at $(0,0)$ (when $t = 0$), at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ (when $t = \frac{\pi}{4}$), and at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ (when $t = \frac{3\pi}{4}$).

In total, we see that the only possible points where f could achieve its global extreme values on E must be $(0,0)$, $(\frac{3}{4}, 0)$, $(1,0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

We have

$$f(0,0) = 2, \qquad f\Big(\frac{3}{4}, 0\Big) = 2\Big(\frac{3}{4}\Big)^2 - 3\Big(\frac{3}{4}\Big) + 2 = \frac{7}{8}, \qquad f(1,0) = 1,$$

$$f\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big) = 1 + 1 - \frac{1}{2} - \frac{6}{\sqrt{2}} + 2 = \frac{7 - 6\sqrt{2}}{2}, \qquad f\Big(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big) = 1 + 1 + \frac{1}{2} + 2 = \frac{9}{2}.$$

Note that since $49 < 72$, taking square roots gives $7 < 6\sqrt{2}$, so that $\frac{7-6\sqrt{2}}{2} < 0$.

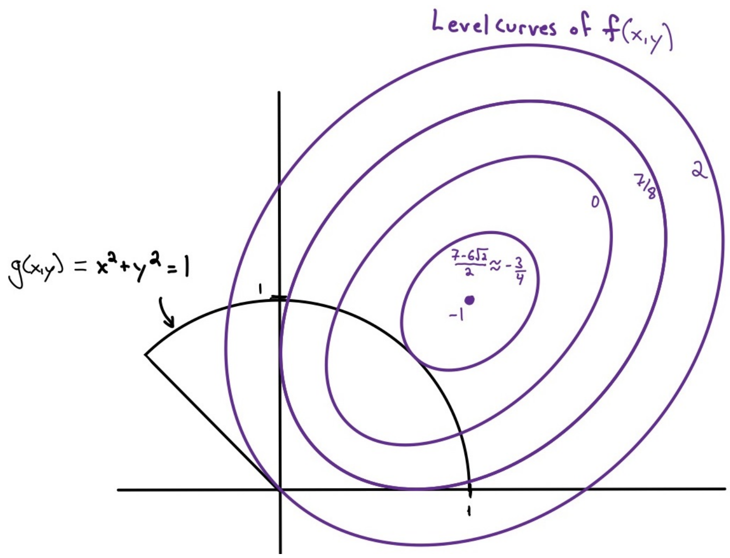
Therefore the global maximum value of f on E is $\frac{9}{2}$ (and occurs at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$). The global minimum value of f on E is $\frac{7-6\sqrt{2}}{2}$ (and occurs at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$).

Constrained Local Extreme Values

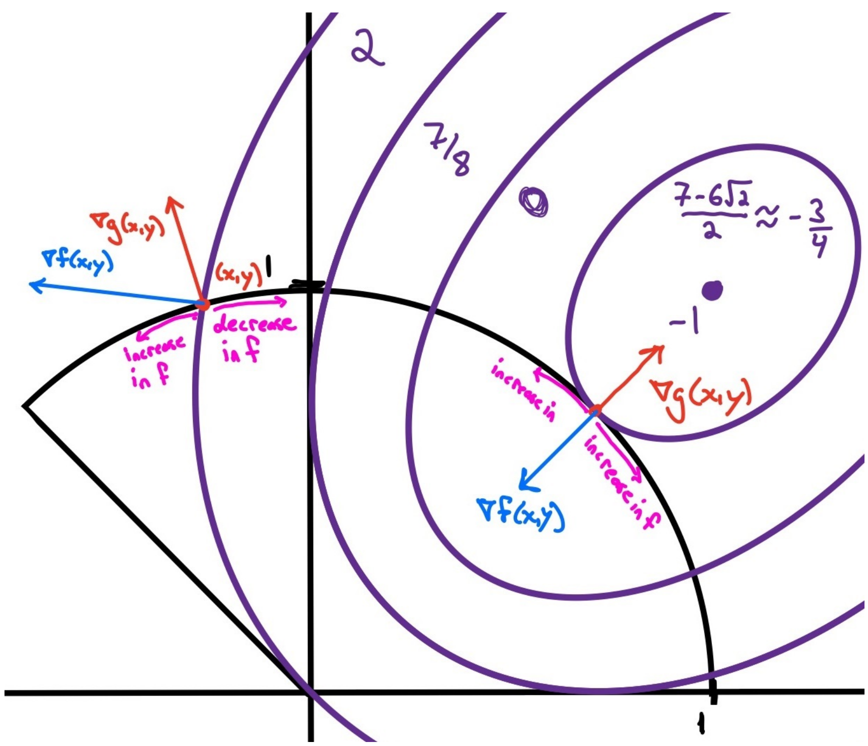
In the previous problem we found the points where $f(\vec{x})$ may have a global extreme value on E when $\vec{x}$ is constrained (or restricted) to each of the curves C_1, C_2, and C_3. Each of these curves is a portion of a curve defined by a single equation: $y = 0$ for C_1, $x + y = 0$ for C_2, and $x^2 + y^2 = 1$ for C_3. We call an equation of this form a **constraint** on the inputs of f.

In practice we are often interested in finding the **constrained extreme values** of a function, which are just the values of the function when the input is subject to a constraint. As we saw in the previous example, attempting to find constrained extreme values through parametrization can get very complicated very quickly, as parameterization is sometimes difficult and usually messy.

We can bring more powerful geometric ideas to bear on the problem through the following observation: constraints can usually be seen as level sets of functions. In the previous example, our analysis on C_3 can be viewed as trying to find the constrained extreme values of f on the level set $g(x,y) = 1$, where $g(x,y) = x^2 + y^2$ (we also have the additional conditions here that $y \geq 0$ and $x \geq -\frac{1}{\sqrt{2}}$, but let's ignore those for now). Below we sketch some level curves of the function $f(x,y) = 2x^2 + 2y^2 - xy - 3y - 3x + 2$, as well as the graph of the constraint $x^2 + y^2 = 1$:



From this picture we expect that for $\vec{a} \in S$, if $f(\vec{a})$ is a local minimum or maximum when compared only against $f(\vec{x})$ for $\vec{x}$ on C_3 near $\vec{a}$ (we call $f(\vec{a})$ a **constrained local extreme value**), then the point $\vec{a}$ should be a point where ∇f is perpendicular to the constraint curve $x^2 + y^2 = 1$. For if ∇f is not perpendicular to the circle at a given point (x, y), then it should be that one can change the input on C_3 slightly in one direction to yield larger values of f, and in the other direction to yield smaller values of f. In the picture below (illustrating a point where the gradient of f is not perpendicular to the constraint), moving along the constraint in the counterclockwise direction from the specified point results in larger values of f, while moving in the clockwise direction results in smaller values of f. (NB: the vectors $\nabla g(x, y)$ and $\nabla f(x, y)$ as shown have the correct directions, but the magnitudes have been adjusted to make the picture easier to parse.)



Thinking of the curve C_3 as the level curve of $g(x, y) = x^2 + y^2 = 1$, this means that $\nabla f(\vec{a}) = \lambda \nabla g(\vec{a})$ for some number λ. The number λ is called a **Lagrange multiplier**. Of course, this reasoning would not detect an extreme value that occurs at one of the endpoints of the curve C_3, but we can check those points manually.

This informal reasoning can be turned into a general result, called the Method of Lagrange Multipliers, which we now state and prove.

Theorem 3 (Lagrange Multipliers)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and suppose that $f, g : \Omega \to \mathbb{R}$ are C^1 on Ω. Let $S = \{\vec{x} \in \Omega \ : \ g(\vec{x}) = c\}$ be the level set of g at height c, and let $\vec{a} \in S$. Assume that $\nabla g(\vec{a}) \neq \vec{0}$, and that the restriction $f : S \to \mathbb{R}$ of f to S has a constrained local extreme value (i.e. a local extreme value when compared only to nearby points on S) at $\vec{a}$. Then there exists $\lambda \in \mathbb{R}$ such that $\nabla f(\vec{a}) = \lambda \nabla g(\vec{a})$.

Proof. The idea of the proof is to note that $\nabla f(\vec{a})$ is orthogonal to every vector that is tangent to S at $\vec{a}$. Because $\nabla g(\vec{a})$ also has this property and the space of vectors orthogonal to S at $\vec{a}$ is one-dimensional, $\nabla f(\vec{a})$ must lie in the span of $\nabla g(\vec{a})$. Here is a sketch[2] of the details.

The condition that g is C^1 near $\vec{a}$ implies that the space T of vectors tangent to S at $\vec{a}$ has dimension $n-1$ and $T^\perp = \operatorname{span}(\nabla g(\vec{a}))$. We show that $\nabla f(\vec{a}) \in T^\perp$.

Suppose that $\vec{u}$ is tangent to S at $\vec{a}$ (i.e. $\vec{u} \in T$). let $\vec{r} : \mathbb{R} \to S$ be a differentiable path such that $\vec{r}(0) = \vec{a}$ and $D\vec{r}(0) = \vec{u}$. Since the restriction $f : S \to \mathbb{R}$ has a local extreme value at $\vec{a}$, the single-variable function $h(t) = f(\vec{r}(t))$ also has a local extreme value at $t = 0$. By the chain rule, we therefore have

$$0 = h'(0) = Df(\vec{r}(0))D\vec{r}(0) = Df(\vec{a})\vec{u} = \nabla f(\vec{a}) \cdot \vec{u}.$$

Therefore $\nabla f(\vec{a})$ is orthogonal to $\vec{u}$. Because $\vec{u} \in T$ was arbitrary,

$$\nabla f(\vec{a}) \in T^\perp = \operatorname{span}(\nabla g(\vec{a})).$$

Therefore there is $\lambda \in \mathbb{R}$ with $\nabla f(\vec{a}) = \lambda \nabla g(\vec{a})$. □

Example 9. Going back to the first example, we want to maximize $f(x, y) = 2x^2 + 2y^2 - xy - 3x - 3y + 2$ subject to the constraint $g(x, y) = x^2 + y^2 = 1$ (and we also require that $x \geq -\frac{1}{\sqrt{2}}$ and $y \geq 0$, but this won't come into play until the end). By the previous theorem, the local extreme values must occur at points $\vec{x}$ satisfying the following system of equations (for some $\lambda \in \mathbb{R}$):

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 1 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} 4x - y - 3 = 2\lambda x \\ 4y - x - 3 = 2\lambda y \\ x^2 + y^2 = 1. \end{cases}$$

Note that the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ exactly captures the condition that $\nabla f(x, y)$ lies in the span of $\nabla g(x, y)$, while the equation $g(x, y) = 1$ restricts our attention to points (x, y) on the constraint.

We solve this system. The first equation on the right can be rewritten as $(4 - 2\lambda)x - y = 3$, and the second equation on the right can be written as $-x + (4 - 2\lambda)y = 3$. Therefore $(4 - 2\lambda)x - y = -x + (4 - 2\lambda)y$, so that $(5 - 2\lambda)(x - y) = 0$. It follows that either $\lambda = \frac{5}{2}$ or $x = y$.

If $\lambda = \frac{5}{2}$ then the first two equations simplify to $x + y = -3$. But the third equation says that we have $\|(x, y)\| = 1$, so that $|x + y| \leq |x| + |y| \leq 2\|(x, y)\| = 2 < 3$, and therefore it is not possible that $x + y = -3$. We conclude that $\lambda \neq \frac{5}{2}$.

[2] I say "sketch" here because the claim that $\dim(T) = n - 1$ is a consequence of the Implicit Function Theorem, which implies that near $\vec{a}$ we can view S as the graph of one variable as a function of the other variables.

Therefore $x = y$, and the third equation yields $x = y = \pm\frac{1}{\sqrt{2}}$, which gives us the points $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. (We could solve for λ here as well, but this is not so important.) These are the only points on the *circle* $x^2 + y^2 = 1$ where f could have a local constrained extreme value. Because we have the additional constraints that $x \geq -\frac{1}{\sqrt{2}}$ and $y \geq 0$, we see that only one of these points is on the portion of the circle that we care about: $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Of course, the endpoints of the curve C_3 could also be places where f has a constrained (global) extreme value on C_3, so we would need to test at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(1, 0)$ as well.

In total, we found that the points on C_3 where f might have a maximum or minimum value are $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $(1, 0)$, which agrees with what we found using the parametrization argument.

Example 10. We want to design a cylindrical bottle to hold 1 liter (1000 cm^3) of water. The base and sides of the bottle are to be metal, and cost \$2 per square centimeter to make, while the top is a high-grade plastic which costs \$3 per square centimeter to make. What dimensions of the bottle will minimize the cost?

We want to minimize the cost function

$$C(r, h) = 2(\pi r^2 + 2\pi rh) + 3(\pi r^2) = 5\pi r^2 + 4\pi rh$$

subject to the constraint

$$\pi r^2 h = 1000.$$

The global minimum value of $C(r, h)$ constrained to $g(r, h) \overset{def}{=} \pi r^2 h = 1000$ (if it exists! More on this below.) must also be a constrained local minimum value, so we can identify where this might occur by using the method of Lagrange multipliers.

The choice (r, h) or radius and height that minimize the cost of the cylinder should satisfy (for some $\lambda \in \mathbb{R}$)

$$\begin{cases} \nabla C(r, h) = \lambda \nabla g(r, h) \\ g(r, h) = 1000 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} 10\pi r + 4\pi h = 2\lambda\pi rh \\ 4\pi r = \lambda\pi r^2 \\ \pi r^2 h = 1000 \end{cases}$$

Multiplying the first equation by r and using the constraint gives us

$$10\pi r^2 + 4\pi rh = 2\lambda\pi r^2 h = 2000\lambda.$$

Similarly, multiplying the second equation by h and using the constraint gives us

$$4\pi rh = \lambda\pi r^2 h = 1000\lambda.$$

Dividing the (new) first equation by 2 and subtracting yields

$$5\pi r^2 - 2\pi rh = 0, \qquad \text{or rather} \qquad r(5r - 2h) = 0.$$

Since $r \neq 0$, we must have $5r = 2h$.

Plugging this into the constraint and solving yields

$$r = 2\sqrt[3]{\frac{50}{\pi}}\ cm, \quad h = 5\sqrt[3]{\frac{50}{\pi}}\ cm.$$

This shows that $(r, h) = \left(2\sqrt[3]{\frac{50}{\pi}}, 5\sqrt[3]{\frac{50}{\pi}}\right)$ is the only point where the restriction of $C(r, h)$ to the constraint $g(r, h) = 1000$ might have a global minimum value.

There is still a subtle issue here: the constraint set described by $g(r, h) = 1000$ is not compact because it is not bounded, and therefore it is not obvious that a global minimum value even exists. To see this, note that we have $h = \frac{1000}{\pi r^2}$, so that the points (r, h) become arbitrarily large in size as $r \to 0+$. (Note that here we only consider $r > 0$ and $h > 0$, as these are suppose to represent the radius and height of a cylinder.)

To get around this difficulty, note that the relationship $h = \frac{1000}{\pi r^2}$ shows that

$$C(r, h) = 5\pi r^2 + 4\pi r h = 5\pi r^2 + \frac{4000}{r} \to \infty \quad \text{as} \quad r \to \infty$$

and, instead writing $r = \sqrt{\frac{1000}{\pi h}}$,

$$C(r, h) = 5\pi r^2 + 4\pi r h = \frac{5000}{h} + 4\pi h\sqrt{\frac{1000}{\pi h}} = \frac{5000}{h} + 40\sqrt{10\pi h} \to \infty \quad \text{as} \quad h \to \infty.$$

Therefore there is some large δ so that if $\|(r, h)\| \geq \delta$, then $C(r, h) \geq C\left(2\sqrt[3]{\frac{50}{\pi}}, 5\sqrt[3]{\frac{50}{\pi}}\right) + 1$. Then the portion of the constraint that lies inside of the closed ball of radius δ,

$$\{(r, h) \ : \ \|(r, h)\| \leq \delta, \pi r^2 h = 1000\},$$

is compact and $C(r, h) \geq C\left(2\sqrt[3]{\frac{50}{\pi}}, 5\sqrt[3]{\frac{50}{\pi}}\right)$ outside of this set.

Because the restriction of $C(r, h)$ attains a global minimum value somewhere in this compact set (by the Extreme Value Theorem), and such a global minimum value on this set must be no greater than $C\left(2\sqrt[3]{\frac{50}{\pi}}, 5\sqrt[3]{\frac{50}{\pi}}\right)$. Because $C(r, h) \geq C\left(2\sqrt[3]{\frac{50}{\pi}}, 5\sqrt[3]{\frac{50}{\pi}}\right) + 1$ if (r, h) lies on the constraint and $\|(r, h)\| \geq \delta$, the global minimum value of $C(r, h)$ in the compact set is actually the global minimum value of $C(r, h)$ on the entire (non-compact) constraint set. Because the global minimum value of $C(r, h)$ must also be a local minimum value, we see that $C\left(2\sqrt[3]{\frac{50}{\pi}}, 5\sqrt[3]{\frac{50}{\pi}}\right)$ must actually be the global minimum value of $C(r, h)$.

Lecture 4: More Constrained Extrema

Learning Objectives:

- Apply the method of Lagrange multipliers with one or several constraints.

We start today with one of the classic computations in multivariable calculus.

Example 11. Let $A \in M_{n\times n}(\mathbb{R})$ be symmetric, and consider the quadratic form $q(\vec{x}) = \vec{x} \cdot (A\vec{x})$. Last quarter we used the Spectral Theorem to determine that the maximum and minimum values of q on the unit sphere $S^{n-1} = \{\vec{x} \ : \ \|\vec{x}\| = 1\}$ are exactly the largest and smallest eigenvalues of A.

We prove this result a second time, now using the method of Lagrange Multipliers. Indeed, we seek to optimize the function $q(\vec{x}) = \vec{x}\cdot(A\vec{x})$ subject to the constraint $g(\vec{x}) = 1$, where $g(\vec{x}) = \|\vec{x}\|^2 = x_1^2+\cdots+x_n^2$. (We use this form for the constraint to make computing the derivative easier.) Because q is continuous on the constraint set $S^{n-1} = \{\vec{x} \ : \ g(\vec{x}) = 1\}$ and S^{n-1} is compact, the Extreme Value Theorem implies that q does indeed attain global extreme values on S^{n-1} (which will also be constrained local extreme values of q).

By the method of Lagrange Multipliers, a point $\vec{x}$ at which q attains a constrained extreme value must satisfy, for some $\lambda \in \mathbb{R}$,

$$\begin{cases} \nabla q(\vec{x}) = \lambda \nabla g(\vec{x}) \\ g(\vec{x}) = 1 \end{cases}$$

Note that since $g(\vec{x}) = x_1^2 + \cdots + x_n^2$, $g_{x_j}(\vec{x}) = 2x_j$ for each $1 \le j \le n$, and therefore $\nabla g(\vec{x}) = 2\vec{x}$.

We will show that $\nabla q(\vec{x}) = 2A\vec{x}$. The computation can be a little tricky, so we will proceed in a way that doesn't involve using too many indices. Fix $1 \le k \le n$, and write $\vec{x} = x_k\vec{e}_k + \vec{v}$, where $\vec{v} = x_1\vec{e}_1 + \cdots + x_{k-1}\vec{e}_{k-1} + x_{k+1}\vec{e}_{k+1} + \cdot + x_n\vec{e}_n$. Note that $\vec{v}$ does not depend on x_k; this will facilitate our computation of q_{x_k}. Indeed, using the fact that A is symmetric we have

$$\begin{aligned} q(\vec{x}) &= (x_k\vec{e}_k + \vec{v}) \cdot (A(x_k\vec{e}_k + \vec{v})) \\ &= x_k^2(\vec{e}_k \cdot (A\vec{e}_k)) + x_k(\vec{e}_k \cdot (A\vec{v})) + x_k(\vec{v} \cdot (A\vec{e}_k)) + \vec{v} \cdot (A\vec{v}) \\ &= x_k^2(\vec{e}_k \cdot (A\vec{e}_k)) + x_k(\vec{e}_k \cdot (A\vec{v})) + x_k((A\vec{v}) \cdot \vec{e}_k) + \vec{v} \cdot (A\vec{v}) \\ &= x_k^2(\vec{e}_k \cdot (A\vec{e}_k)) + 2x_k(\vec{e}_k \cdot (A\vec{v})) + \vec{v} \cdot (A\vec{v}), \end{aligned}$$

so that

$$\begin{aligned} q_{x_k}(\vec{x}) &= 2x_k(\vec{e}_k \cdot (A\vec{e}_k)) + 2(\vec{e}_k \cdot (A\vec{v})) + 0 \\ &= \vec{e}_k \cdot (2A(x_k\vec{e}_k)) + \vec{e}_k \cdot (2A\vec{v}) \\ &= \vec{e}_k \cdot (2A(x_k\vec{e}_k + \vec{v})) \\ &= \vec{e}_k \cdot (2A\vec{x}). \end{aligned}$$

Because $q_{x_k}(\vec{x})$ is the k-th entry of $\nabla q(\vec{x})$, and $\vec{e}_k \cdot (2A\vec{x})$ is the k-th entry of $2A\vec{x}$, we conclude that $\nabla q(\vec{x}) = 2A\vec{x}$.

Therefore the equation $\nabla q(\vec{x}) = \lambda \nabla g(\vec{x})$ simplifies to $2A\vec{x} = \lambda(2\vec{x})$, or rather $A\vec{x} = \lambda\vec{x}$. Because the equation $g(\vec{x}) = 1$ implies that $\|\vec{x}\|^2 = 1$ (and therefore $\vec{x} \neq \vec{0}$), it follows that such a point $\vec{x}$ we have that $\vec{x}$ is an eigenvector and $q(\vec{x}) = \vec{x} \cdot (A\vec{x}) = \vec{x} \cdot (\lambda\vec{x}) = \lambda\|\vec{x}\|^2 = \lambda$. Therefore we conclude that the global maximum value of q on S^{n-1} is the largest eigenvalue of q, and the global minimum value of q on S^{n-1} is the smallest eigenvalue of q.

Lagrange Multipliers with Multiple Constraints

The version of the method of Lagrange Multipliers that we established last time is great when we are attempting to optimize a function $f : \mathbb{R}^n \to \mathbb{R}$ constrained to a subset that is the level set of a single C^1 function $g : \mathbb{R}^n \to \mathbb{R}$. In practice we are often led to consider the case where there are multiple constraints on the inputs of f, say $g_1(\vec{x}) = c_1,\ g_2(\vec{x}) = c_2,\ \ldots\ , g_m(\vec{x}) = c_m$.

If we assume that the set $S = \{\vec{x}\ :\ g_i(\vec{x}) = c_i \text{ for } i = 1, \ldots, m\}$ is appropriately non-degenerate (in the sense that the space of vectors tangent to S at a point has the "correct" dimension), then we can generalize the method of Lagrange multipliers to identify points where f has constrained local extreme values on S. We can think of S as the intersection of the level sets $S_i = \{\vec{x}\ :\ g_i(\vec{x}) = c_i\}$, each of which is an $(n-1)$-dimensional set that locally resembles $\mathbb{R}^{n-1}$ (in the same way that a curve locally resembles $\mathbb{R}^1$, and a surface locally resembles $\mathbb{R}^2$). This exact notion is made more precise in a course in differential geometry, but hopefully the intuition here makes sense.

One fact from differential geometry that we need is that if the vectors $\nabla g_1(\vec{a}), \ldots, \nabla g_m(\vec{a})$ form a linearly independent set, then the level sets $S_1, \ldots, S_m$ intersect[3] "nicely", in the sense that the vectors that are tangent to S at $\vec{a}$ are exactly the vectors that are tangent to each of $S_1, \ldots, S_m$ at $\vec{a}$.

In this case, it follows that if T is the set of vectors tangent to S at $\vec{a}$, then

$$T = \Big(\operatorname{span}(\nabla g_1(\vec{a}), \ldots, \nabla g_m(\vec{a}))\Big)^{\perp}.$$

But by repeating part of the proof of the single-constraint case of the Method of Lagrange Multipliers, we see that $\nabla f(\vec{a}) \in T^{\perp}$. It therefore follows that $\nabla f(\vec{a}) \in \operatorname{span}(\nabla g_1(\vec{a}), \ldots, \nabla g_m(\vec{a}))$. We summarize this reasoning in the following theorem.

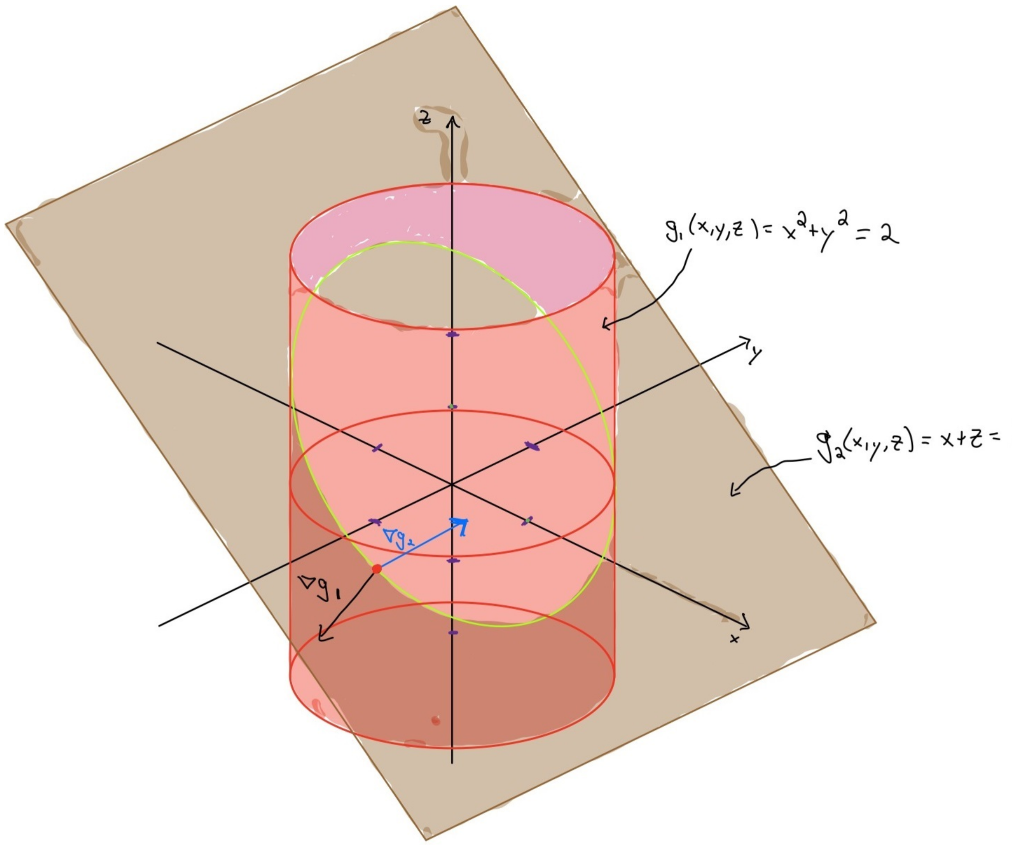
Theorem 4 (Lagrange Multipliers - Multiple Constraints)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open, and assume that $f, g_1, \ldots, g_m : \Omega \to \mathbb{R}$ are C^1 on Ω. Let $S = \{\vec{x} \in \Omega\ :\ g_i(\vec{x}) = c_i,\ i = 1, \ldots, m\}$ and $\vec{a} \in S$. Assume that $\nabla g_1(\vec{a}), \ldots, \nabla g_m(\vec{a})$ forms a linearly independent set, and that the restriction $f : S \to \mathbb{R}$ of f to S has a constrained local extreme value (i.e. a local extreme value when compared to nearby points on S) at $\vec{a}$. Then there exists $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

$$\nabla f(\vec{a}) = \lambda_1 \nabla g_1(\vec{a}) + \cdots + \lambda_m \nabla g_m(\vec{a}).$$

Example 12. Find the extreme values of $f(x, y, z) = x + y + z$ on the intersection of the cylinder $x^2 + y^2 = 2$ and the plane $x + z = 1$.

We are attempting to find the extreme values of f subject to the constraints $g_1(x, y, z) = 2$ and $g_2(x, y, z) = 1$, where $g_1(x, y, z) = x^2 + y^2$ and $g_2(x, y, z) = x + z$. Because the first constraint restricts (x, y, z) to the cylinder of radius $\sqrt{2}$ centered on the z-axis, and the second constraint restricts (x, y, z) to the plane $x + z = 1$, imposing both of these constraints restricts (x, y, z) to the intersection of the cylinder and the plane.

[3] Alas, this is another consequence of the Implicit Function Theorem!



Note that this curve is compact, and therefore the continuous function f does indeed have global extrema on this curve (which must be constrained local extrema). We will apply the multiple-constraint version of the method of Lagrange multipliers.

Note that

$$\nabla g_1(x,y,z) = \begin{bmatrix} 2x \\ 2y \\ 0 \end{bmatrix} \quad \text{and} \quad \nabla g_2(x,y,z) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent for (x, y, z) satisfying $x^2 + y^2 = 2$ and $x + z = 1$, since if

$$\nabla g_1(x,y,z) + c_2 \nabla g_2(x,y,z) = \begin{bmatrix} 2c_1 x + c_2 \\ 2c_1 y \\ c_2 \end{bmatrix} = \vec{0}$$

then we must have $c_2 = 0$, and therefore (since at least one of x or y is nonzero) we have $c_1 = 0$ as well. Therefore the Lagrange Multipliers Theorem for multiple constraints applies, and at the points where f might have local extreme values (and therefore global extreme values) must satisfy, for scalars $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\begin{cases} \nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z) \\ g_1(x,y,z) = 2 \\ g_2(x,y,z) = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 1 = \lambda_1 2x + \lambda_2 \\ 1 = \lambda_1 2y \\ 1 = \lambda_2 \\ x^2 + y^2 = 2 \\ x + z = 1 \end{cases}$$

At such points we must have $\lambda_2 = 1$, so that $2\lambda_1 x = 0$, or rather $\lambda_1 x = 0$. Because $\frac{1}{2} = \lambda_1 y$, we have $\frac{1}{4} = 0^2 + \left(\frac{1}{2}\right)^2 = \lambda_1^2 x^2 + \lambda_1^2 y^2 = \lambda_1^2(x^2 + y^2) = 2\lambda_1^2$ by the fourth equation, so that $\lambda_1 = \pm\frac{1}{2\sqrt{2}}$. If $\lambda_1 = \frac{1}{2\sqrt{2}}$, then we have $x = 0$ (so that $z = 1$) and $y = \sqrt{2}$. If $\lambda_1 = -\frac{1}{2\sqrt{2}}$, then we have $x = 0$ (so that $z = 1$) and $y = -\sqrt{2}$.

Therefore the points where f might have extreme values on the intersection of the plane and the sphere are $(0, \sqrt{2}, 1)$ and $(0, -\sqrt{2}, 1)$. At these points we have $f(0, \sqrt{2}, 1) = 1 + \sqrt{2}$ and $f(0, -\sqrt{2}, 1) = 1 - \sqrt{2}$, so that the global maximum value of f is $1 + \sqrt{2}$, and the global minimum is $1 - \sqrt{2}$.

Lecture 5: Riemann Sums

Learning Objectives:

- Describe the geometric intuition of the Riemann integral in terms of dimension-appropriate volume.
- Use partitions and sample points to construct a Riemann sum for a function over a box.
- Describe integrability and integrals in terms of limits of Riemann sums.

Now that we have completed our treatment of multivariable differential calculus, we turn to multivariable integral calculus. Just as in single-variable calculus there are two important (and related) notions of integration: one geometric (related to areas, volumes, etc.), and the other dynamic (related to an appropriate notion of antidifferentiation). Both of these notions are important, and they are related in single-variable calculus by the Fundamental Theorem of Calculus. We will spend the rest of this quarter generalizing these ideas (including the Fundamental Theorem of Calculus) to the multivariable setting. We start by understanding integration from a geometric point of view.

What can you use from Single-Variable Integral Calculus?

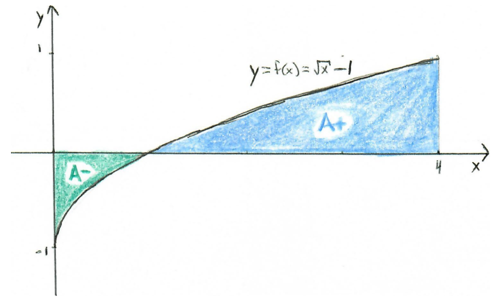
This is not a single-variable calculus course, and we will therefore utilize ideas and techniques that were established in your single-variable integral calculus. Some topics that you should feel free to use (and therefore might find helpful to review) are:

- standard antiderivative formulas (power rule, $\frac{1}{x}$, trigonometric functions, inverse trigonometric functions, exponential functions)
- antidifferentiation techniques (substitution for indefinite integrals, substitution for definite integrals, integration by parts, partial fraction decomposition, trigonometric integrals, trigonometric substitution)
- The Fundamental Theorem of Calculus
- improper integrals

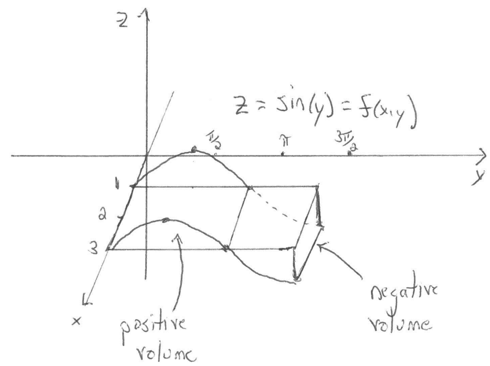
Don't hesitate to reach out to Prof. Peterson or the TA if you need help reviewing any of these ideas!

Meaning of Multiple Integrals

For a scalar-valued function $f : \mathbb{R} \to \mathbb{R}$ of a single-variable, the integral of f over an interval $[a, b]$ was understood to measure the (signed) area of the region enclosed between the graph of f and the x-axis over the interval $[a, b]$, with points where $f(x) > 0$ contributing "positive area" and points where $f(x) < 0$ contributing "negative area":



This generalizes to higher-dimensions in the same way. For example, if Ω is a region in $\mathbb{R}^2$ and if $f : \mathbb{R}^2 \to \mathbb{R}$, then the integral of f over Ω should measure the (3-dimensional) "signed volume" of the region in $\mathbb{R}^3 = \mathbb{R}^{2+1}$ between the graph of f and the xy-plane over the region Ω, with points where $f(x, y) > 0$ contributing "positive volume" and points where $f(x, y) < 0$ contributing "negative volume":



If Ω is a region in $\mathbb{R}^n$ and if $f : \mathbb{R}^n \to \mathbb{R}$, then the integral of f over Ω should measure the "signed $(n + 1)$-volume" of the region in $\mathbb{R}^{n+1}$ between the graph of f and the region Ω (thought of as lying in the subspace $\mathbb{R}^n$ of $\mathbb{R}^{n+1}$), with points where $f(\vec{x}) > 0$ contributing "positive $(n + 1)$-volume" and points where $f(\vec{x}) < 0$ contributing "negative $(n + 1)$-volume".

At this point we cannot draw any more pictures, but we can note that the notion of n-volume we will use here should be compatible with the notion of n-volume that we worked with when we studied parallelotopes in MATH 291-2. To leverage this, we will build our notion of integration using boxes (the higher-dimensional analogues of rectangles), for which we can compute the appropriate notion of volume in the usual way.

Riemann Integration

To formalize the idea of what the integral of f should be, we build up some fundamental notions.

Boxes

Definition 4. A **box** $B \subset \mathbb{R}^n$ is a set of the form

$$B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \overset{def}{=} \{(x_1, x_2, \ldots, x_n) \;:\; x_i \in [a_i, b_i] \text{ for } i = 1, \ldots, n\}.$$

Here we assume that $a_i \leq b_i$ for each $i = 1, \ldots, n$.

Remark 2. A box $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is exactly the set of points $\vec{x} \in \mathbb{R}^n$ where each coordinate x_i lies in the interval $[a_i, b_i]$. For example:

(a) A box in $\mathbb{R}$ is just an interval $[a, b]$.

(b) A box in $\mathbb{R}^2$ is a rectangular-shaped region with sides parallel to the coordinate axes. Such a box has the form $[a, b] \times [c, d]$, which consists of points (x, y) where $x \in [a, b]$ and $y \in [c, d]$.

(c) A box in $\mathbb{R}^3$ is a solid box (in the ordinary usage of the word) with sides parallel to the coordinate planes. Such a box has the form $[a, b] \times [c, d] \times [e, f]$, which consists of points (x, y, z) where $x \in [a, b]$ and $y \in [c, d]$ and $z \in [e, f]$.

Remark 3. Note that a box $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$ is just a (shifted) parallelotope with sides parallel to $\vec{e}_1, \ldots, \vec{e}_n$, since

$$\begin{aligned} B &= \{(x_1, x_2, \ldots, x_n) \ : \ x_i \in [a_i, b_i] \text{ for } i = 1, \ldots, n\} \\ &= \{(a_1 + t_1(b_1 - a_1), a_2 + t_2(b_2 - a_2), \ldots, a_n + t_n(b_n - a_b) \ : \ t_i \in [0, 1] \text{ for } i = 1, \ldots, n\} \\ &= (a_1, \ldots, a_n) + \{t_1(b_1 - a_1)\vec{e}_1 + t_2(b_2 - a_2)\vec{e}_2 + \cdots + t_n(b_n - a_n)\vec{e}_n \ : \ t_i \in [0, 1] \text{ for } i = 1, \ldots, n\} \\ &= (a_1, \ldots, a_n) + E((b_1 - a_1)\vec{e}_1, (b_2 - a_2)\vec{e}_2, \ldots, (b_n - a_n)\vec{e}_n). \end{aligned}$$

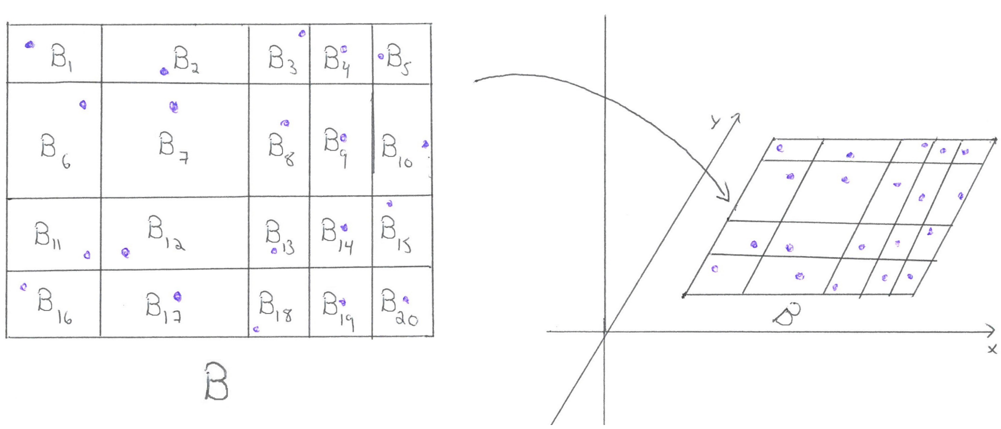
Because the box $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a (shifted by $(a_1, \ldots, a_n)$) parallelotope in $\mathbb{R}^n$ determined by the vectors $(b_1 - a_1)\vec{e}_1, \ldots, (b_n - a_n)\vec{e}_n$, we apply our notion of n-volume from last quarter to make the following definition.

Definition 5. The ***n*-volume** of a box $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$ is defined as

$$\text{Vol}_n(B) \stackrel{def}{=} \text{Vol}_n(E((b_1 - a_1)\vec{e}_1, (b_2 - a_2)\vec{e}_2, \ldots, (b_n - a_n)\vec{e}_n)) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n).$$

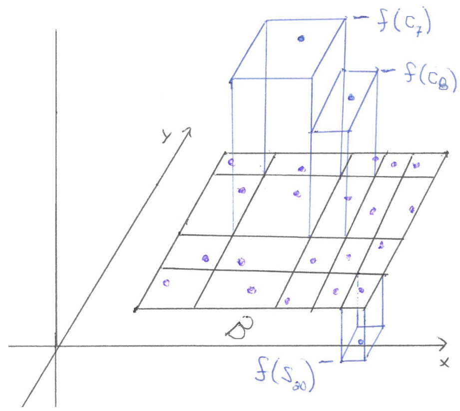
Partitions and Riemann Sums

The integral of $f : \mathbb{R}^n \to \mathbb{R}$ over a box B is defined analogously to how the integral is defined in single-variable calculus: as a limit of Riemann sums. A Riemann sum can be thought of as an approximation of the integral of f over B obtained by first **partitioning** (i.e. splitting) B into smaller boxes $B_1, B_2, \ldots, B_M$, choosing one **sample point** $\vec{c}_1, \ldots, \vec{c}_M$ in each B_i. (The sample points $\vec{c}_i$ are shown in the picture below in purple.) We allow (only) the boundaries of $B_1, B_2, \ldots, B_M$ to overlap.



We denote the collection of smaller boxes as $\mathcal{P}$, and the choice of sample points as $\mathcal{C}$.

Given the partition $\mathcal{P}$ of B and the choice of sample points $\mathcal{C}$, we approximate the signed $(n+1)$-volume between the graph of f and B_i as $f(\vec{c}_i)\text{Vol}_n(B_i)$. Here $f(\vec{c}_i)$ should be thought of as the "height" of a box in $\mathbb{R}^{n+1}$ with base B_i. (Here we are allowing the "height" to be negative, hence the quotes.)



Each partition $\mathcal{P}$ of B and choice of sample points $\mathcal{C}$ determines a **Riemann sum** $R(f, \mathcal{P}, \mathcal{C})$, which is an approximation for the signed $(n+1)$-volume between the graph of f and the box B:

$$R(f, \mathcal{P}, \mathcal{C}) \stackrel{def}{=} \sum_i f(\vec{c}_i)\text{Vol}_n(B_i).$$

Let $\|\mathcal{P}\|$ denote the maximum edge-length over all of the smaller boxes that make up the partition $\mathcal{P}$. In a perfect world, as $\|\mathcal{P}\| \to 0$ (i.e. as the maximum edge-length of the boxes created by the partition $\mathcal{P}$ approach 0), we would hope that $R(f, \mathcal{P}, \mathcal{C})$ would approach "the" integral of f over B. Indeed, we will actually *define* the integral of f in terms of this limiting process.

Definition 6. Let B be a box in $\mathbb{R}^n$, and let $f : B \to \mathbb{R}$. Then we define the **(Riemann) integral of** f **over** B to be

$$\int_B f(\vec{x})\, dV_n(\vec{x}) \stackrel{def}{=} \lim_{\|\mathcal{P}\| \to 0} R(f, \mathcal{P}, \mathcal{C}),$$

provided that the limit on the right exists. In this case we say that f is **(Riemann) integrable** over B.

Remark 4. Your book (and we) will refer to the type of integral described in the previous definition as a **multiple integral**, signifying that we are integrating a function over a multidimensional set as a limit of Riemann sums. When $n = 2$ it is standard to say **double integral** instead of multiple integral, and when $n = 3$ it is standard to say **triple integral** instead of multiple integral. These are standard names for these special cases that you will see "out in the wild" and in your textbook, but they all refer to the integral of a function over a box as a limit of Riemann sums.

Remark 5. The notation $dV_n(\vec{x})$ is intended to denote an "infinitesimal" version of $\text{Vol}_n(B_i)$ from the Riemann sums, and so should be considered the infinitesimal n-volume of a box in B containing $\vec{x}$. This is merely a suggestive notational device, but will be convenient for framing various results in easy-to-remember ways.

When $n = 2$ we may write $dA(x, y)$ instead of $dV_2(x, y)$ (because 2-volume is just area), and when $n = 3$ we may write $dV(x, y, z)$ instead of $dV_3(x, y, z)$ (because 3-volume is just the traditional notion of volume).

Notation 1. When there is no ambiguity about which variables we are integrating in, then we may just write $f dV_n$ instead of $f(\vec{x})dV_n(\vec{x})$ for brevity.

Remark 6. If $B \subset \mathbb{R}^n$ is a box with $\mathrm{Vol}_n(B) > 0$, and if $f : B \to \mathbb{R}$ is unbounded (in the sense that its set of outputs is not bounded), then f is not integrable on B. In other words, a necessary precondition for a function to be integrable on a box is that it have bounded outputs!

To see why, suppose (towards a contradiction) that f is integrable on B and let

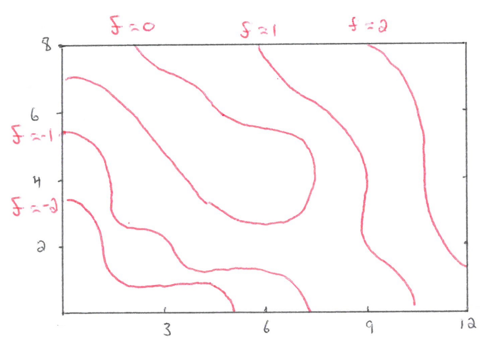
$$L = \int_B f\, dV_n = \lim_{\|\mathcal{P}\| \to 0} R(f, \mathcal{P}, \mathcal{C}).$$

Choose $\delta > 0$ such that if $\|\mathcal{P}\| < \delta$ then $|L - R(f, \mathcal{P}, \mathcal{C})| < 1$ regardless of the choice $\mathcal{C}$ of sample points. Let $\mathcal{P}$ be a partition of B into boxes $B_1, \ldots, B_N$ with nonzero n-volume, such that $\|\mathcal{P}\| < \delta$. Then there must be some i_0 such that the outputs of f on B_{i_0} are unbounded, for if not then for each $i = 1, \ldots, N$ there is $M_i \geq 0$ with $|f(\vec{x})| \leq M_i$ for each $\vec{x} \in B_i$. But then $|f(\vec{x})| \leq \max(M_1, \ldots, M_N)$ for each $\vec{x} \in B$, contrary to our assumption that f is unbounded. Let $\mathcal{C}$ be a choice of sample points $\vec{c}_i \in B_i$. Because f is unbounded on B_{i_0} and $\mathrm{Vol}_n(B_{i_0}) > 0$, we can choose $\vec{c}_i^{\,*}$ such that $|f(\vec{c}_{i_0}) - f(\vec{c}_{i_0}^{\,*})| > 2/(\mathrm{Vol}_n(B_{i_0}))$. Then

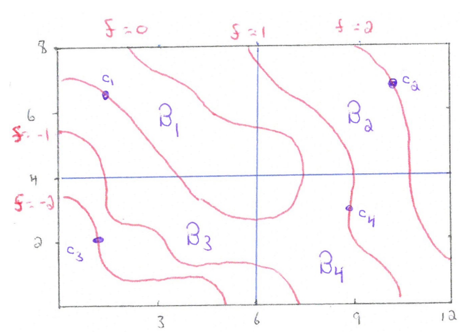
$$\begin{aligned}
2 &< |(f(\vec{c}_{i_0}) - f(\vec{c}_{i_0}^{\,*}))\mathrm{Vol}_n(B_{i_0})| \\
&= |(f(\vec{c}_{i_0}) - f(\vec{c}_{i_0}^{\,*}))\mathrm{Vol}_n(B_{i_0}) + L - \sum_{i \neq i_0} f(\vec{c}_i)\mathrm{Vol}_n(B_i) - L + \sum_{i \neq i_0} f(\vec{c}_i)\mathrm{Vol}_n(B_i)| \\
&= |L - R(f, \mathcal{P}, \mathcal{C}^*) - (L - R(f, \mathcal{P}, \mathcal{C}))| \\
&\leq |L - R(f, \mathcal{P}, \mathcal{C}^*)| + |L - R(f, \mathcal{P}, \mathcal{C})| \\
&< 1 + 1 = 2,
\end{aligned}$$

a contradiction. Therefore whenever we assume that a function f is integrable on a box B, we may assume that f is bounded on B.

Example 13. Consider the function $f(x, y)$ (some of the level curves of which are given below) over the box $B = [0, 12] \times [0, 8]$. Let's use Riemann sums to estimate $\int_B f\, dA$.



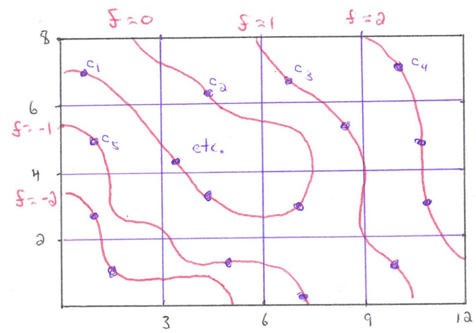
First we partition B into 4 equal-sized rectangles, and choose a sample point in each rectangle:



The Riemann sum for f given these choices of partition $\mathcal{P}$ and sample points $\mathcal{C}$ is

$$\begin{aligned} R(f,\mathcal{P},\mathcal{C}) &= f(\vec{c}_1)\text{Area}(B_1) + f(\vec{c}_2)\text{Area}(B_2) + f(\vec{c}_3)\text{Area}(B_3) + f(\vec{c}_4)\text{Area}(B_4) \\ &= 0(24) + 2(24) + -2(24) + 1(24) \\ &= 24. \end{aligned}$$

Let's now make the rectangles smaller, say by partitioning R into 16 equal sizes rectangles, with sample points as indicated:



The Riemann sum for $f(x,y)$ given these choices of partition $\mathcal{P}'$ and sample points $\mathcal{C}'$ is

$$\begin{aligned} R(f,\mathcal{P}',\mathcal{C}') &= 0(6) + 0(6) + 1(6) + 2(6) + (-1)(6) + 0(6) + 1(6) + 2(6) \\ &\quad +(-2)(6) + 0(6) + 0(6) + 2(6) + (-2)(6) + (-1)(6) + (-1)(6) + 1(6) \\ &= 12. \end{aligned}$$

Remark 7. Although we will not use it formally for any proofs, it can sometimes be helpful to a definition of integrability that is easier to use. What follows is an equivalent definition of the Riemann integral of a function (due to Darboux). To this end, suppose that B is a box in $\mathbb{R}^n$ and that $f : B \to \mathbb{R}$ has bounded outputs. Let $\mathcal{P}$ be a partition of B. For a box B_i given by the partition, let m_i be the largest real number for which $m_i \leq f(\vec{x})$ for all $\vec{x} \in B_i$ and let M_i be the smallest real number for which $f(\vec{x}) \leq M_i$ for all $\vec{x} \in B_i$. (Note that m_i and M_i will be the minimum and maximum values of f on B_i *if f actually achieves maximum and minimum values on B_i*.) We call

$$U(f,\mathcal{P}) \overset{def}{=} \sum_i M_i \text{Vol}_n(B_i) \quad \text{and} \quad L(f,\mathcal{P}) \overset{def}{=} \sum_i m_i \text{Vol}_n(B_i)$$

the **upper sum** and **lower sum** of f on B relative to the partition $\mathcal{P}$. The key point here is that because of how m_i and M_i were chosen,

$$L(f,\mathcal{P}) \leq R(f,\mathcal{P},\mathcal{C}) \leq U(f,\mathcal{P})$$

for every possible choice of sample points $\mathcal{C}$.

We can then state the following formal definition of integrability:

Definition 7. Suppose that B is a box and that $f : B \to \mathbb{R}$ has bounded outputs. Then we say that f is **integrable on** B if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of B such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

In other words, f is integrable over B if for each $\epsilon > 0$ there is a partition $\mathcal{P}$ such that the difference between the upper sum of f (which should be an *overestimate* of the integral of f over B) and the lower sum of f (which should be an *underestimate* of the integral of f over B) is less than ϵ.

One way to interpret this is that, for each $\epsilon > 0$ there is a partition $\mathcal{P}$ of B such that the graph of f over B is covered by the boxes $E_i = B_i \times [m_i, M_i]$, and that

$$\sum \mathrm{Vol}_{n+1}(E_i) = \sum (M_i - m_i)\mathrm{Vol}_n(B_i) = \sum M_i \mathrm{Vol}_n(B_i) - \sum m_i \mathrm{Vol}_n(B_i) = U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

Lecture 6: Integrability

Learning Objectives:

- Establish properties of the integral by using the definition of integrability.
- Determine when a subset of $\mathbb{R}^n$ has measure zero.
- Determine when a function is integrable by inspecting its sets of discontinuities.

Last time we built up the notion of the integral of a function $f : B \to \mathbb{R}$ on a box $B \subset \mathbb{R}^n$ using Riemann sums. This definition can be used directly to compute the integrals of only very simple functions.

Using the Definition of Integrability

Example 14. Let $B \subset \mathbb{R}^n$ be a box, and suppose $f : B \to \mathbb{R}$ is constant (i.e. $f(\vec{x}) = c$ for some constant $c \in \mathbb{R}$). Then f is integrable on B and

$$\int_B f \, dV_n = c\mathrm{Vol}_n(B).$$

To prove this, suppose that $\mathcal{P}$ is a partition of B and $\mathcal{C}$ is a choice of sample points for $\mathcal{P}$. Then we have

$$R(f, \mathcal{P}, \mathcal{C}) = \sum_i f(\vec{c}_i)\mathrm{Vol}_n(B_i) = \sum_i c\mathrm{Vol}_n(B_i) = c\sum_i \mathrm{Vol}_n(B_i) = c\mathrm{Vol}_n(B),$$

where the last equality follows from the fact that B is the union of the boxes $B_1, \ldots, B_M$ and these boxes only overlap on their boundaries (and each 'face' of the boundary of one of the B_i can be viewed as a parallelotope determined by a set of n vectors that includes $\vec{0}$, and therefore has n-volume 0). Therefore we have

$$\lim_{\|\mathcal{P}\| \to 0} R(f, \mathcal{P}, \mathcal{C}) = \lim_{\|\mathcal{P}\| \to 0} c\mathrm{Vol}_n(B) = c\mathrm{Vol}_n(B).$$

It follows that f is integrable on B and $\displaystyle\int_B f(\vec{x}) \, dV_n = c\mathrm{Vol}_n(B)$.

We will have additional (and much better) ways to compute the integrals of more interesting functions in a couple days. The definition of the integral is incredibly useful for proving abstract properties of the integral. To illustrate this, we note that we can prove familiar linearity properties and inequalities for the integral with relative ease.

Theorem 5 (Properties of the Integral)**.** Suppose that f and g are both integrable on a box $B \subset \mathbb{R}^n$. Then the the following properties hold.

1. $f+g$ is also integrable on B and
$$\int_B (f+g)\,dV_n = \int_B f\,dV_n + \int_B g\,dV_n.$$
2. For $c \in \mathbb{R}$, cf is also integrable on B and
$$\int_B cf\,dV_n = c\int_B f\,dV_n.$$
3. If $f(\vec{x}) \le g(\vec{x})$ for all $\vec{x} \in B$, then
$$\int_B f\,dV_n \le \int_B g\,dV_n.$$
4. $|f|$ is also integrable on B and
$$\left|\int_B f\,dV_n\right| \le \int_B |f|\,dV_n.$$

Proof. You will prove parts 2., 3., and 4. on your homework (in the special case where $n=2$, but your proof will work for all n). We illustrate how these types of arguments go by proving part 1.

For 1., let $\mathcal{P}$ be a partition of B and let $\mathcal{C}$ be a choice of sample points for $\mathcal{P}$. Then we have
$$\begin{aligned} R(f+g,\mathcal{P},\mathcal{C}) &= \sum_i (f(\vec{c}_i)+g(\vec{c}_i))\mathrm{Vol}_n(B_i) \\ &= \sum_i f(\vec{c}_i)\mathrm{Vol}_n(B_i) + \sum_i g(\vec{c}_i)\mathrm{Vol}_n(B_i) \\ &= R(f,\mathcal{P},\mathcal{C}) + R(g,\mathcal{P},\mathcal{C}). \end{aligned}$$
Because $\lim\limits_{\|\mathcal{P}\|\to 0} R(f,\mathcal{P},\mathcal{C}) = \int_B f(\vec{x})\,dV_n$ and $\lim\limits_{\|\mathcal{P}\|\to 0} R(g,\mathcal{P},\mathcal{C}) = \int_B g(\vec{x})\,dV_n$, we have
$$\lim_{\|\mathcal{P}\|\to 0} R(f+g,\mathcal{P},\mathcal{C}) = \lim_{\|\mathcal{P}\|\to 0}\Big[R(f,\mathcal{P},\mathcal{C}) + R(g,\mathcal{P},\mathcal{C})\Big] = \int_B f(\vec{x})\,dV_n + \int_B g(\vec{x})\,dV_n,$$
so that $f+g$ is integrable on B and
$$\int_B (f+g)\,dV_n = \int_B f\,dV_n + \int_B g\,dB_n.$$
□

Integrable Functions

Now that we know the basic properties of the integral, we can turn our attention to the nuanced question of which functions are actually integrable. We saw in our opening example that constant functions are integrable. It may not surprise you to learn that all continuous functions are integrable.

Theorem 6. Let $B \subset \mathbb{R}^n$ be a box, and suppose that $f : B \to \mathbb{R}$ is continuous. Then f is integrable on B.

Proof. We prove this using the the more precise definition of Riemann integrability from the previous lecture, but we will also need a fact from analysis about continuous functions that we did not prove last quarter. In particular, we will use the fact that since f is continuous on B and B is compact, then f is **uniformly continuous** on B in the sense that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $\vec{x}, \vec{y} \in B$ with $\|\vec{x} - \vec{y}\| < \delta$, it follows that $|f(\vec{x}) - f(\vec{y})| < \epsilon$. Note that this is stronger than mere continuity, as we are able to choose δ here depending on ϵ but *independent* of $\vec{x}$ and $\vec{y}$.

We now proceed with the proof. To avoid trivialities, assume that $\mathrm{Vol}_n(B) > 0$. Let $\epsilon > 0$. Because f is continuous on B and B is compact, f is uniformly continuous on B. Choose $\delta > 0$ such that for every $\vec{x}, \vec{y} \in B$ with $\|\vec{x} - \vec{y}\| < \delta$, it follows that $|f(\vec{x}) - f(\vec{y})| < \frac{\epsilon}{2\mathrm{Vol}_n(B)}$. Let $\mathcal{P}$ be a partition of B with $\|\mathcal{P}\| < \frac{\delta}{\sqrt{n}}$, and let B_i be any box that is part of the partition $\mathcal{P}$ of B. Because B_i is compact and f is continuous on B_i, the Extreme Value Theorem implies that there exist $\vec{y}, \vec{z} \in B_i$ with

$$f(\vec{y}) = m_i \overset{def}{=} \min_{\vec{x} \in B_i} f(\vec{x}) \quad \text{and} \quad f(\vec{z}) = M_i \overset{def}{=} \max_{\vec{x} \in B_i} f(\vec{x}).$$

Note that because $\vec{y}, \vec{z} \in B_i$ and the maximum length of an edge of B_i is $\|\mathcal{P}\| < \frac{\delta}{\sqrt{n}}$, we have that

$$\|\vec{y} - \vec{z}\| = \sqrt{(y_1 - z_1)^2 + \cdot + (y_n - z_n)^2} < \sqrt{\frac{\delta^2}{n} + \cdots + \frac{\delta^2}{n}} = \delta,$$

and therefore $0 \leq M_i - m_i = f(\vec{z}) - f(\vec{y}) < \frac{\epsilon}{2\mathrm{Vol}_n(B)}$. It follows that

$$\begin{aligned}
U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_i M_i \mathrm{Vol}_n(B_i) - \sum_i m_i \mathrm{Vol}_n(B_i) \\
&= \sum_i (M_i - m_i) \mathrm{Vol}_n(B_i) \\
&\leq \sum_i \frac{\epsilon}{2\mathrm{Vol}_n(B)} \mathrm{Vol}_n(B_i) \\
&= \frac{\epsilon}{2\mathrm{Vol}_n(B)} \sum_i \mathrm{Vol}_n(B_i) \\
&= \frac{\epsilon}{2\mathrm{Vol}_n(B)} \mathrm{Vol}_n(B) \\
&= \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Therefore f is integrable on B. □

Measure Zero

We can weaken the assumption of continuity a little if our function is bounded (in the sense that the set of outputs is bounded) and only discontinuous on a "small" set. Here, "small" means that the set has **measure zero**, in the sense that it lies inside of another set that we know has n-volume as small as we'd like. Because we only know how to measure the volume of boxes (and at least overestimate the volumes of unions of boxes by adding the volumes of the boxes), we are led to the following definition.

Definition 8. Let $A \subseteq \mathbb{R}^n$. Say that A has **measure zero** if for every $\epsilon > 0$ there exists a (finite or countably infinite[4]) collection of boxes $B_1, B_2, B_3, \ldots \subset \mathbb{R}^n$ such that

$$A \subset \bigcup_i B_i = B_1 \cup B_2 \cup B_3 \cup \cdots \quad \text{and} \quad \sum_i \mathrm{Vol}_n(B_i) < \epsilon.$$

Note that if there are indeed an infinite number of boxes, then this sum is actually a series.

In other words, a subset of $\mathbb{R}^n$ has measure zero if it can be covered by a finite or (countably) infinite collection of boxes, where we can take the total volume of these boxes to be arbitrarily small.

Example 15. The interval $[0,2]$ in $\mathbb{R}$ does not have measure zero, because $\mathrm{Vol}_1([0,2]) = 2$, and therefore the sum of 1-volumes of boxes that cover $[0,2]$ must be at least 2 (and not 0).

But the set $I = \{(x,0) \ : \ x \in [0,2]\}$ in $\mathbb{R}^2$ does have measure zero, since for $\epsilon > 0$ we note that $I \subseteq [0,2] \times [0, \frac{\epsilon}{3}]$, and

$$\mathrm{Vol}_2\Big([0,2] \times \Big[0, \frac{\epsilon}{3}\Big]\Big) = \frac{2\epsilon}{3} < \epsilon.$$

These results confirm what we already know from intuition: the interval $[0,2]$ has length (i.e. 1-volume) 2, but has area (i.e. 2-volume) 0. It also illustrates an important point: whether a geometric object (like a line segment) has measure zero depends on the space in which it lives. For example, a (solid) square would have measure zero in $\mathbb{R}^3$, but would have positive measure in $\mathbb{R}^2$.

Example 16. The set of rational numbers $\mathbb{Q}$ has measaure zero in $\mathbb{R}$. To prove this, we need to use the fact that the rational numbers are countably infinite[5]. That is, it is possible to write the elements of $\mathbb{Q}$ in an infinite list $r_1, r_2, r_3, \ldots$. Let $\epsilon > 0$. Tor each $k \in \mathbb{N}$, set $B_k = [r_k, r_k]$. Then B_k is a box and $\mathrm{Vol}_1(B_k) = r_k - r_k = 0$, so that $\mathbb{Q} \subset \bigcup_k B_k$ and $\sum_i \mathrm{Vol}_1(B_k) = 0 < \epsilon$.

The fact that $\mathbb{Q}$ has measure zero in $\mathbb{R}$ is even more remarkable once one considers the surprising fact that $\mathbb{Q}$ is **dense** in $\mathbb{R}$, in the sense that for every $x \in \mathbb{R}$ and every $\epsilon > 0$ there is $r \in \mathbb{Q}$ with $|r - x| < \epsilon$. That is, for every $x \in \mathbb{R}$ we can find a rational number that is as close to x as we wish. In this sense, the elements of $\mathbb{Q}$ are "everywhere" in $\mathbb{R}$, but somehow manage to have measure zero!

Example 17. The xz-plane P in $\mathbb{R}^3$ has measure zero. To see why, let $\epsilon > 0$ and choose

$$B_i = [-i, i] \times [0, \frac{\epsilon}{4i^2 3^i}] \times [-i, i] \quad \text{for } i = 1, 2, 3, \ldots.$$

Then if $(x, 0, z) \in P$, i greater than the maximum of $|x|$ and $|z|$ we have $(x, 0, z) \in B_i$, so that

$$P \subseteq \bigcup_i B_i.$$

[4]A set S is **countably infinite** if there is a bijection between S and the natural numbers $\mathbb{N}$. The natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, and any infinite subset of these sets is countably infinite. The real numbers $\mathbb{R}$ (indeed, any interval of the form (a,b) where $a < b$) is **uncountably** infinite, in the sense that it is infinite but admits no bijection with $\mathbb{N}$. Talk to me in office hours if you want to learn more about various types of infinity!

[5]There are many proofs of this result. Ask me about it in office hours!

On the other hand, $\mathrm{Vol}_3(B_i) = (2i)(2i)\frac{\epsilon}{4i^2 3^i} = \frac{\epsilon}{3^i}$ and therefore (using the formula for the sum of a convergent geometric series, where here the first term is $a = \frac{\epsilon}{3}$ and the ratio $r = \frac{1}{3}$ satisfies $|r| < 1$)

$$\sum_{i=1}^{\infty} \mathrm{Vol}_3(B_i) = \sum_{i=1}^{\infty} \frac{\epsilon}{3^i} = \frac{\epsilon/3}{1-\frac{1}{3}} = \frac{\epsilon}{2} < \epsilon.$$

Therefore P has measure zero.

The following theorem is indispensable for identifying interesting sets of measure zero.

Theorem 7 (Measure Zero)**.** Let $A \subseteq \mathbb{R}^n$. Then A has measure zero if

(Subset of a Set of Measure Zero) $A \subseteq B$ where $B \subseteq \mathbb{R}^n$ has measure zero, or

(Finite Union of Measure Zero) $A = \bigcup_i B_i$, where $B_1, \dots, B_k \subseteq \mathbb{R}^n$ each have measure zero, or

(Image Lower Dimensional Set) $A = \vec{f}(B)$, where $B \subseteq \mathbb{R}^m$ for $m < n$ and $\vec{f} : B \to \mathbb{R}^n$ is C^1, or

(Non-Degenerate Level Set) A is a level set of some C^1 function $g : \mathbb{R}^n \to \mathbb{R}$ with $\nabla g(\vec{x}) \neq \vec{0}$ for every $\vec{x} \in A$.

Proof. You will prove the first part on your homework.

For the second part, let $\epsilon > 0$ and choose, for each B_i, a (finite or countable) collection of boxes that covers B_i and has total measure less than $\frac{\epsilon}{k}$. Because the union of a finite collection of finite or countable sets is either finite or countable, combining these collections of boxes produces a finite or countable collection of boxes that covers A and has total measure less than $k \cdot \frac{\epsilon}{k} = \epsilon$.

The third part is a special case of **Sard's Theorem**, a powerful result in differential geometry.

The fourth part follows from the third part and another powerful result (this time from multivariable differential calculus, but usually proved in an analysis course) called the **Implicit Function Theorem**, which says that, under the hypotheses of the fourth part, for each $\vec{a} \in A$ there is a ball $B_\delta(\vec{a})$ centered at $\vec{a}$ such that $A \cap B_\delta(\vec{a})$ can actually be written as the graph of one of the variables as a C^1 function of the other variables (e.g. in the case where z can be written as a function of x and y, then the points $(x, y, z) \in A \cap B_\delta(\vec{a})$ have the form $(x, y, z(x, y))$ for a C^1 function $z(x, y)$, so that $A \cap B_\delta(\vec{a})$ is the image of the C^1 function $(x, y) \mapsto (x, y, z(x, y))$. The same comments apply if the points in $A \cap B_\delta(\vec{a})$ have the form $(x, y(x, z), z)$ or $(x(y, z), y, z)$). □

Remark 8. The second conclusion of the Measure Zero Theorem actually holds if $A = \bigcup_i B_i$, where $B_1, B_2, B_3, \ldots \subseteq \mathbb{R}^n$ is a countable collection of sets with measure zero. The proof is even largely the same, with some adjustments needed to ensure that the total measure of the combined collection of boxes has measure less than ϵ.

The previous theorem is very powerful. Here are some immediate consequences.

Example 18. The image of any C^1 path $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ has measure zero if $n \geq 2$.

Example 19. For $A, B, C \neq 0$, the ellipsoid described by $\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$ has measure zero, since it is the level set of $g(x, y, z) = \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}$ with $\nabla g(x, y, z) \neq \vec{0}$ everywhere on the set.

Indeed, this same argument shows that all of the standard quadric surfaces have measure zero. The only care that must be taken is to handle the double-cone, but this can be viewed as the union of the upper-half described by $z = \sqrt{x^2 + y^2}$ where $x, y \neq 0$ (which is the level set of a C^1 function $g(x, y, z) = z - \sqrt{x^2 + y^2}$ on $\mathbb{R}^3 - \{(0, 0, z) \ : \ z \in \mathbb{R}\}$), the lower-half cone $z = -\sqrt{x^2 + y^2}$ where $x, y \neq 0$, and the single point $(0, 0, 0)$ (which has measure zero).

The concept of measure zero offers us a complete characterization of which functions are integrable (as defined last time), and which are not.

Theorem 8 (Lebesgue's Criterion)**.** Let $B \subset \mathbb{R}^n$ be a box, and let $f : B \to \mathbb{R}$ be bounded. Then f is integrable on B if, and only if, the set of points where f is discontinuous has measure zero.

The proof of this result is technical, and beyond the scope of this course.

Lecture 7: Iterated Integrals

Learning Objectives:

- Integrate continuous functions over bounded sets with nice boundaries.
- Compute multiple integrals using Fubini's Theorem.

Integration on More General Sets

For us, one major application of Lebesgue's Criterion will be to allow us to integrate continuous functions on sets that are not boxes. The process here is simple: we simply choose a box that contains the set that we wish to integrate over, extend the domain of the function we want to integrate to be 0 outside of the set we care about, and then integrate this extension function over the box. We make this more precise with the following definition.

Definition 9. Let $E \subseteq \Omega \subseteq \mathbb{R}^n$ with E bounded, let $B \subset \mathbb{R}^n$ be a box containing E, and suppose that ∂E has measure zero. Assume $f : \Omega \to \mathbb{R}$ is bounded on E and continuous throughout E (except possibly on a set of measure zero), and define the **zero extension** of f from E to B by

$$f^{ext} : B \to \mathbb{R}, \quad f^{ext}(\vec{x}) \overset{def}{=} \begin{cases} f(\vec{x}) & \text{if } \vec{x} \in E, \\ 0 & \text{if } \vec{x} \notin E. \end{cases}$$

Note that f^{ext} is only possibly discontinuous on ∂E (which has measure zero), and therefore f^{ext} is integrable on B. We define the **integral of f on E** to be

$$\int_E f \, dV_n \overset{def}{=} \int_B f^{ext} \, dV_n.$$

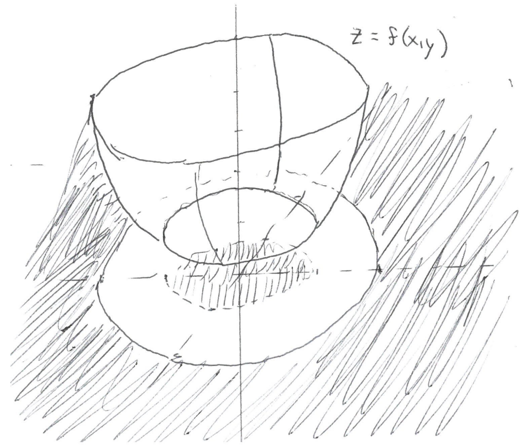
Example 20. For example, if $E = \{(x, y) \in \mathbb{R}^2 \ : \ 1 \leq \sqrt{x^2 + y^2} \leq 2\}$ is the annulus centered at $(0, 0)$ with inner radius 1 and outer radius 2 in $\mathbb{R}^2$, then

$$\int_E (x^2 + y^2) \, dA(x, y) = \int_{[-2,2]\times[-2,2]} f^{ext}(x, y) \, dA(x, y),$$

where $f(x, y) = x^2 + y^2$ and

$$f^{ext}(x, y) = \begin{cases} x^2 + y^2 & \text{if } 1 \leq \sqrt{x^2 + y^2} \leq 2, \\ 0 & \text{otherwise} \end{cases}$$

is integrable over the rectangle $[-2, 2] \times [-2, 2]$ because f^{ext} is only discontinuous along the circles of radius 1 and 2 centered at $(0, 0)$ (and since a circle, which is the C^1 image of a path in $\mathbb{R}^2$, has measure zero).



Remark 9. As a convention, if we integrate a function f over a set B and if f is defined by a formula such that $f(\vec{x})$ is actually undefined for finitely many $\vec{x} \in B$, then we will artificially define $f(\vec{x}) = 0$ at those points. This will allow us to avoid a proliferation of technical arguments around defining integrals in practice.

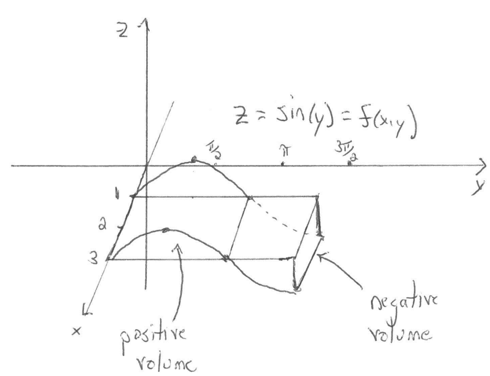
Iterated Integrals and Fubini's Theorem

As mentioned the other day, there are only a limited number of functions that we can actually integrate *by hand* using the definition of the integral as a limit of Riemann sums. For more interesting examples, we will need a way to approach this computation without resorting to Riemann sums. One fruitful approach is to use **iterated** single-variable integrals, which are related to multiple integrals via a powerful result known as **Fubini's Theorem**. Because the notion of iterated integrals can involve complicated notation, we motivate their use and interpretation through a concrete example.

Example 21. Compute[6]

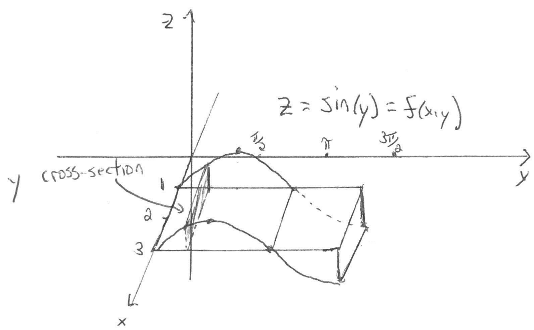
$$I = \iint\limits_{[1,3]\times[0,\frac{3\pi}{2}]} \sin(y)\, dA(x,y).$$

Below we draw the graph of $z = f(x,y) = \sin(y)$.



[6]When integrating over subsets $\mathbb{R}^2$, it is standard to write $\iint$ instead of $\int$ to reflect the fact that we are integrating over a two-dimensional set. This is analogous to replacing dV_2 with dA. Similarly, for integrals over sets in $\mathbb{R}^3$ it is standard to write a triple integral sign $\iiint$ instead of a single integral sign. We will continue to use a single integral sign in "dimension agnostic" settings.

To motivate the use of iterated integrals, consider that the y-sections of this region (i.e. the intersection of this region with planes perpendicular to the y-axis at $(0, y, 0)$) are rectangles with width $3 - 1 = 2$ and height $\sin(y)$ (Note that this height can be negative!).



Therefore,

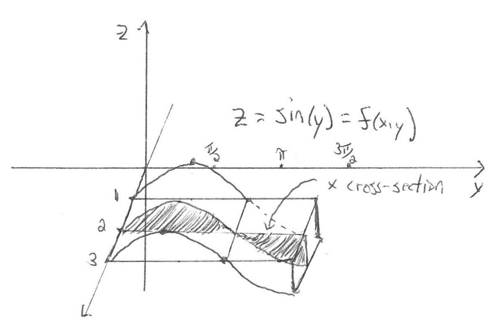
$$\text{Area of cross-section at } y: \qquad \int_1^3 \sin(y)dx = 2\sin(y).$$

We might expect that the (signed) volume of the region between the graph of f and the box $[1,3]\times[0,\frac{3\pi}{2}]$ (as a subset of the xy-plane) to be obtained by 'adding up' these signed areas over all the possible values of y by integrating in y. In other words, we expect that

$$\begin{aligned} I &= \int_0^{\frac{3\pi}{2}} [\text{Area of cross-section at } y]dy \\ &= \int_0^{\frac{3\pi}{2}} \int_1^3 \sin(y)dxdy \\ &= \int_0^{\frac{3\pi}{2}} 2\sin(y)dy \\ &= -2\cos(y)\Big|_0^{\frac{3\pi}{2}} \\ &= 2. \end{aligned}$$

On the other hand, we could have used the x-sections of the region (i.e. the intersection of this region with planes $x = x_0$) to compute the area. For this problem,

$$\text{Area of cross-section at } x = \int_0^{\frac{3\pi}{2}} \sin(y)dy = -\cos(y)\Big|_0^{\frac{3\pi}{2}} = 1.$$



Note that (in this problem) all of these cross-sections are identical, so it isn't surprising that they have the same (signed) area. 'Adding up' these areas over all possible values of x with an integral suggests that

$$\begin{aligned} I &= \int_1^3 [\text{Area of cross-section at } x] dx \\ &= \int_1^3 \int_0^{\frac{3\pi}{2}} \sin(y) dy dx \\ &= \int_1^3 1 dx \\ &= x\Big|_1^3 \\ &= 3 - 1 = 2, \end{aligned}$$

which is the same answer that we got in our first computation!

In the previous example, we actually did not prove that the double integral I of $f(x, y) = \sin(y)$ over $[1, 3] \times [0, \frac{3\pi}{2}]$ was equal to 2. We *did* show that each **iterated integral**

$$\int_1^3 \left[\int_0^{3\pi/2} f(x, y)\, dy \right] dx \quad \text{and} \quad \int_0^{3\pi/2} \left[\int_1^3 f(x, y)\, dx \right] dy$$

existed is equal to 2, and we have strong geometric intuition to suggest that the value of these two iterated integrals ought to agree with I.

It is indeed true that, under suitable mild conditions on the integrand $f(x_1, \ldots, x_n)$, the multiple integral $\int_B f\, dV_n$ of f over a box B can be computed as an iterated integral involving n single-variable integrals (one in each of the variables $x_1, x_2, \ldots, x_n$). Moreover—and this is the crucial fact—the order in which we compute these single-variable integrals does not matter. This result is known as **Fubini's Theorem**, and we state it in full generality below.

Theorem 9 (Fubini)**.** Let $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a box in $\mathbb{R}^n$, let $\Omega \subseteq \mathbb{R}^n$ with $B \subseteq \Omega$, and suppose that $f : \Omega \to \mathbb{R}$ is integrable on B. Assume that for each $1 \le i \le n$ and $x_j \in [a_j, b_j]$ (for $j \neq i$), the single-variable function

$$g_i : [a_i, b_i] \to \mathbb{R}, \quad g_i(t) = f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$$

is integrable on $[a_i, b_i]$. Then for any ordering $i_1, \ldots, i_n$ of the numbers $1, \ldots, n$, we have

$$\int_B f(x_1, \ldots, x_n)\, dV_n(x_1, \ldots, x_n) = \int_{a_{i_1}}^{b_{i_1}} \int_{a_{i_2}}^{b_{i_2}} \cdots \int_{a_{i_n}}^{b_{i_n}} f(x_1, x_2, \ldots, x_n)\, dx_{i_n} \cdots dx_{i_2} dx_{i_1}.$$

Remark 10. The conclusion about iterated integrals is just to say that you can compute the iterated integral of f over B in any order that you wish. For example, if $n = 3$ and $f(x, y, z)$ satisfies the hypotheses of Fubini's Theorem on the box $B = [a, b] \times [c, d] \times [e, g]$, then the conclusion of Fubini's Theorem is that each of the **six** iterated integrals

$$\int_a^b \int_c^d \int_e^g f(x, y, z) dz dy dx \quad \text{and} \quad \int_c^d \int_a^b \int_e^g f(x, y, z) dz dx dy \quad \text{and}$$

$$\int_e^g \int_c^d \int_a^b f(x,y,z)dxdydz \quad \text{and} \quad \int_c^d \int_e^g \int_a^b f(x,y,z)dxdzdy \quad \text{and}$$
$$\int_e^g \int_a^b \int_c^d f(x,y,z)dydxdz \quad \text{and} \quad \int_a^b \int_e^g \int_c^d f(x,y,z)dydzdx$$
exist and are equal to the multiple (in this case, triple) integral $\iiint\limits_B f(x,y,z)\,dV(x,y,z)$.

A full proof of Fubini's Theorem is beyond the scope of the course, but we can give a proof in the case $n=2$ to illustrate one argument. Part of the conclusion of Fubini's Theorem is the existence of the iterated integrals on the right-hand-side. The key idea of the proof is to approximate each single-variable integral with a one-variable Riemann sum, and then note that the resulting expression (for a given iterated integral) gives a Riemann sum for f on B. By choosing the one-variable Riemann partitions to be "fine" enough, we can ensure that the resulting Riemann sum for f on B is as close as we'd like to the integral of f over B. The argument is a bit technical, but we will give the full details here as a partial advertisement for the sorts of estimates you might expect to prove in a course in real analysis.

Proof of Fubini's Theorem when $n=2$. Write $B=[a,b]\times[c,d]$, and define
$$g(x) \overset{def}{=} \int_c^d f(x,y)\,dy, \quad x\in[a,b].$$
To avoid trivialities we will assume that $a<b$ and $c<d$. Note that g is defined because $y\mapsto f(x,y)$ is integrable on $[c,d]$ for every fixed $x\in[a,b]$ by assumption. We will show that g is integrable on $[a,b]$ and that
$$\int_a^b g(x)\,dx = \int_B f(x,y)\,dA(x,y),$$
This will show that
$$\int_a^b \int_c^d f(x,y)\,dydx = \iint\limits_B f(x,y)\,dA(x,y),$$
which is our desired conclusion.

Let $\epsilon>0$. Choose $\rho>0$ such that if $\mathcal{P}$ is a partition of B with $\|\mathcal{P}\|<\rho$ and $\mathcal{C}$ is any choice of sample points for $\mathcal{P}$, then $|R(f,\mathcal{P},\mathcal{C}) - \iint\limits_B f(x,y)\,dA| < \frac{\epsilon}{2}$. Suppose we have a partition $\{[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]\}$ of $[a,b]$ such that $(x_k-x_{k-1})<\rho$ for each $k=1,\ldots,n$, and a choice of sample points $c_k\in[x_{k-1},x_k]$ for $k=1,\ldots,n$. Then the corresponding Riemann sum for g over $[a,b]$ is
$$\begin{aligned} &R(g,\{[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]\},\{c_1,\ldots,c_n\}) \\ &= \sum_{k=1}^n g(c_k)\mathrm{Vol}_1([x_{k-1},x_k]) \\ &= \sum_{k=1}^n \int_c^d f(c_k,y)\,dy\cdot(x_k-x_{k-1}). \end{aligned}$$

Now choose a partition $\{[y_0,y_1],[y_1,y_2],\ldots,[y_{m-1},y_m]\}$ of $[c,d]$ such that $(y_\ell-y_{\ell-1})<\rho$ for each $\ell=1,\ldots,m$ and, for each $k=1,\ldots,n$,
$$\left|\int_c^d f(c_k,y)\,dy - \sum_{\ell=1}^m f(c_k,y_\ell)(y_\ell-y_{\ell-1})\right| < \frac{\epsilon}{2(b-a)}.$$

Then consider the partition

$$\mathcal{P} \stackrel{def}{=} \{B_{k,\ell} = [x_{k-1}, x_k] \times [y_{\ell-1}, y_\ell] \ : \ k = 1, \ldots, n, \ \ell = 1, \ldots, m\}$$

of B and the choice of sample points

$$\mathcal{C} \stackrel{def}{=} \{\vec{c}_{k,\ell} = (c_k, y_\ell) \ : \ k = 1, \ldots, n, \ \ell = 1, \ldots, m\}.$$

Note that because $(x_k - x_{k-1}) < \rho$ and $(y_\ell - y_{\ell-1}) < \rho$ for each k and ℓ, $\|\mathcal{P}\| < \rho$. We also have

$$R(f, \mathcal{P}, \mathcal{C}) = \sum_{k=1}^{n} \sum_{\ell=1}^{m} f(\vec{c}_{k,\ell}) \mathrm{Vol}_2(B_{k,\ell}) = \sum_{k=1}^{n} \sum_{\ell=1}^{m} f(c_k, y_\ell)(y_\ell - y_{\ell-1})(x_k - x_{k-1}).$$

Then we have

$$\begin{aligned}
&\left| R(g, \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}, \{c_1, \ldots, c_n\}) - \iint_B f(x, y)\, dA \right| \\
&= \left| \sum_{k=1}^{n} \int_c^d f(c_k, y)\, dy \cdot (x_k - x_{k-1}) - \iint_B f(x, y)\, dA \right| \\
&= \left| \sum_{k=1}^{n} \int_c^d f(c_k, y)\, dy \cdot (x_k - x_{k-1}) - R(f, \mathcal{P}, \mathcal{C}) + R(f, \mathcal{P}, \mathcal{C}) - \iint_B f(x, y)\, dA \right| \\
&\leq \left| \sum_{k=1}^{n} \int_c^d f(c_k, y)\, dy \cdot (x_k - x_{k-1}) - R(f, \mathcal{P}, \mathcal{C}) \right| + \left| R(f, \mathcal{P}, \mathcal{C}) - \iint_B f(x, y)\, dA \right| \\
&< \left| \sum_{k=1}^{n} \int_c^d f(c_k, y)\, dy \cdot (x_k - x_{k-1}) - \sum_{k=1}^{n} \sum_{\ell=1}^{m} f(x_k, y_\ell)(y_\ell - y_{\ell-1})(x_k - x_{k-1}) \right| + \frac{\epsilon}{2} \\
&= \left| \sum_{k=1}^{n} \left(\int_c^d f(c_k, y)\, dy - \sum_{\ell=1}^{m} f(x_k, y_\ell)(y_\ell - y_{\ell-1}) \right) (x_k - x_{k-1}) \right| + \frac{\epsilon}{2} \\
&\leq \sum_{k=1}^{n} \left| \int_c^d f(c_k, y)\, dy - \sum_{\ell=1}^{m} f(c_k, y_\ell)(y_\ell - y_{\ell-1}) \right| (x_k - x_{k-1}) + \frac{\epsilon}{2} \\
&\leq \sum_{k=1}^{n} \frac{\epsilon}{2(b-a)} (x_k - x_{k-1}) + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2(b-a)} \sum_{k=1}^{n} (x_k - x_{k-1}) + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2(b-a)} (b - a) + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, we have shown that for each $\epsilon > 0$ there is $\rho > 0$ such that if ☺ $= \{[x_0, x_1], \ldots, [x_{n-1}, x_n]\}$ is a partition of $[a, b]$ with $\|$☺$\| < \rho$ and if ☹ $= \{c_1, \ldots, c_n\}$ is a choice of sample points for ☺, then

$$|R(g, ☺, ☹) - \iint_B f(x, y)\, dA| < \epsilon.$$

This shows that $\lim_{\|☺\|\to 0} R(g, ☺, ☹) = \iint_B f(x,y)\,dA$, so that g is integrable on $[a,b]$ and

$$\int_a^b \int_c^d f(x,y)\,dydx = \int_a^b g(x)\,dx = \iint_B f(x,y)\,dA(x,y),$$

as desired.

The same argument shows that

$$h(y) \stackrel{def}{=} \int_a^b f(x,y)\,dx, \quad y \in [c,d]$$

is defined and integrable on $[c,d]$, and that

$$\int_c^d \int_a^b f(x,y)\,dxdy = \int_c^d h(y)\,dy = \iint_B f(x,y)\,dA(x,y),$$

which completes the proof. □

Note that the assumption that f be integrable on B necessitates (via Lebesgue's Criterion for Riemann Integrability) that the set of points in B at which f is discontinuous must have measure zero. The integrand f satisfies the technical "integrable in each coordinate separately" condition if any of the following (stronger) hypotheses are satisfied:

(i) if f is continuous on B, or

(ii) if the set of discontinuities of f intersects each line parallel to one of the coordinate axes in at most finitely many points (this is the assumption in the book's version of the theorem), or

(iii) if the iterated integrals exist (this follows from the "integrable in each coordinate separately" condition, but could also be taken as an assumption).

There are other sufficient conditions for f to satisfy the conclusion of Fubini's Theorem, but (i) and (ii) are the ones that will play the biggest role in this course.

Example 22. Let's compute the double integral of $f(x,y) = ye^{xy}$ over the rectangle $B = [0,1] \times [0,2]$.

We apply Fubini's Theorem, first integrating in x and then in y, to obtain

$$\iint_B ye^{xy} dA(x,y) = \int_0^2 \int_0^1 ye^{xy} dxdy = \int_0^2 e^y - 1 dy = e^2 - 3.$$

On the other hand, if we first try to integrate in y and then in x, we get (by integration by parts)

$$\iint_B ye^{xy} dA(x,y) = \int_0^1 \int_0^2 ye^{xy} dydx = \int_0^1 \Big(\frac{y}{x} - \frac{1}{x^2}\Big) e^{xy}\Big|_0^2 dx = \int_0^1 \frac{(2x-1)e^{2x}+1}{x^2}\,dx.$$

This last integral is not approachable through any of the standard integration techniques used in Calculus II! This illustrates a good point about double-integration: although the multiple integral of a function over a box can be written as an *theoretically* by applying Fubini's theorem, the order of integration can be very important from a *practical* standpoint!

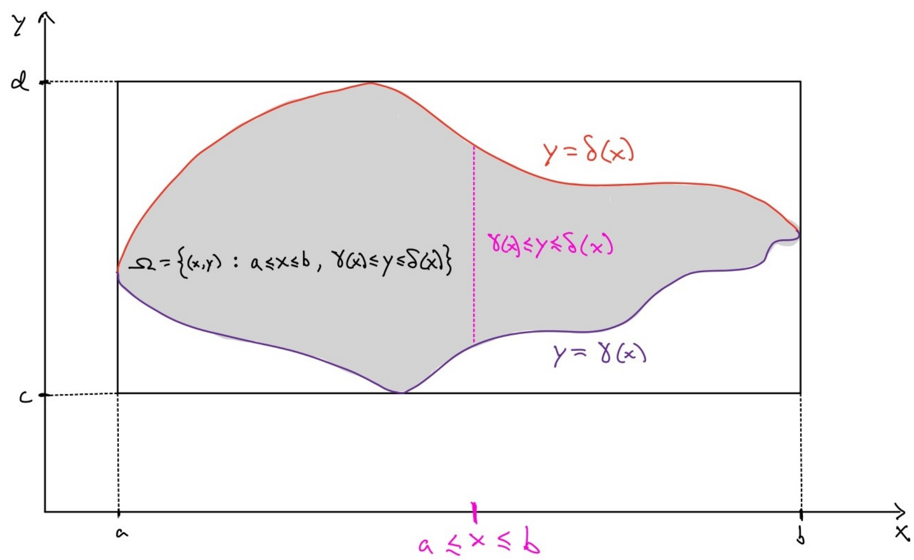
Lecture 8: Double Integrals

Learning Objectives:

- Evaluate a double integral by setting up an appropriate iterated integral.
- Change the order or integration in an iterated integral.

Today we investigate the typical considerations when evaluating double-integrals of functions over general regions. We start by noting that if $f(x, y)$ is continuous in a bounded region Ω and if the extension of f to some box containing Ω satisfies the hypotheses of Fubini's Theorem, then setting up the iterated integral(s) we will use to evaluate $\iint_\Omega f\,dA$ involves an analysis of how to describe Ω (or pieces of Ω) using nested inequalities.

For example, suppose that $[a, b] \times [c, d]$ is the smallest box containing Ω, and that Ω is as shown in the following picture:



Then to represent $\iint_\Omega f(x, y)\,dA$ using an iterated integral in the order $dydx$, we first note that $[a, b]$ is (what your book calls the) **shadow**[7] of Ω onto the x-axis, in the sense that it is the collection of all values of x for which there is y such that $(x, y) \in \Omega$.

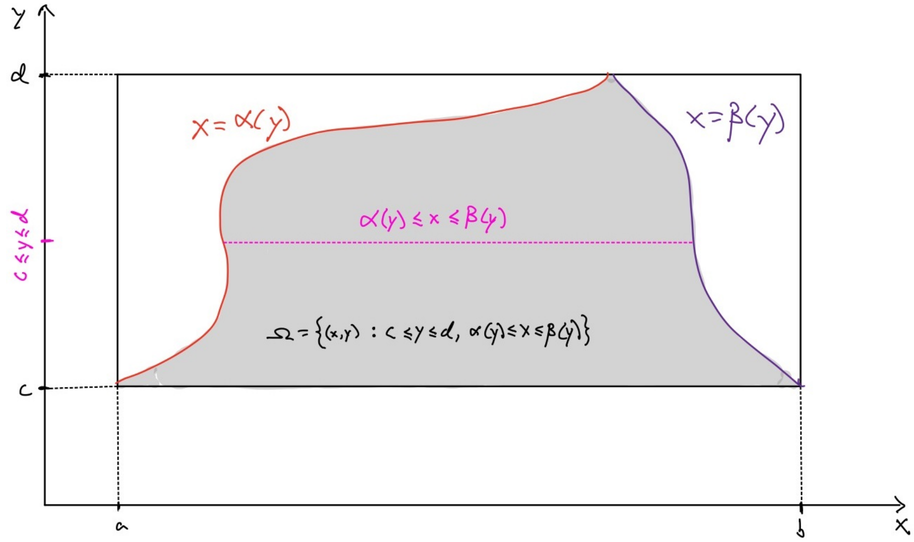
The inner integral $\int_c^d f^{ext}(x, y)\,dy$ is then the integral of $f^{ext}(x, y)$ along the line segment from (x, c) to (x, d). Of course, because $f^{ext}(x, y) = 0$ if $(x, y) \notin \Omega$, we really only need to integrate $f^{ext}(x, y)$ over the the subset of this line segment for which $(x, y) \in \Omega$. The key observation (for Ω as pictured above) is that if we can think about Ω as the region lying between the graphs of two functions $y = \gamma(x)$ and $y = \delta(x)$, then for each $a \le x \le b$ this subset consists of points of the form (x, y) with $\gamma(x) \le y \le \delta(x)$. Therefore we can write

$$\iint\limits_\Omega f(x, y)\,dA(x, y) = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x, y)\,dydx.$$

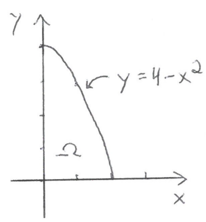
[7] It might make more sense to call this the **projection** of Ω onto the x-axis, but your book assumes that you do not now linear algebra. Foolish book!

Similarly, if Ω can be described by the (nested) inequalities $c \le y \le d$ and $\alpha(y) \le x \le \beta(y)$ (i.e. if we can think about Ω as the region lying between the graphs of two functions $x = \alpha(y)$ and $x = \beta(y)$ over the interval $c \le y \le d$), then

$$\iint_\Omega f(x,y)\,dA(x,y) = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\,dxdy.$$



Example 23. Integrate $f(x,y) = xy$ over the region Ω in the first quadrant below the curve $y = 4 - x^2$.

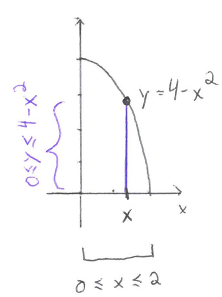


Let's solve this problem two ways. First, we'll set up the interated integral so that x is the "outer-variable". Our double integral is

$$\iint_\Omega xy\,dA(x,y) = \int_0^2 \Big[\int_{\text{lower bound for y at x}}^{\text{upper bound for y at x}} xy\,dy \Big] dx.$$

At each x, y runs from 0 to $4 - x^2$. So, we have

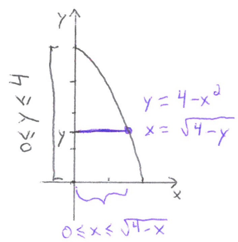
$$\iint_\Omega xy\,dA(x,y) = \int_0^2 \int_0^{4-x^2} xy\,dydx.$$



It may look a little strange to have an x in the bounds of an integral. However, it is in the bounds of the *inner* intergral, and in the context of the inner integral, x is a constant. Of course, the bounds of the *outer* integral cannot depend on x or y. We finish the computation:

$$\iint_{\Omega} xy\, dA(x,y) = \int_0^2 \int_0^{4-x^2} xy\, dydx = \int_0^2 \frac{1}{2}x(4-x^2)^2 dx = -\frac{1}{12}(4-x^2)^3\Big|_0^2 = 0 + \frac{1}{12}4^3 = \frac{16}{3}.$$

To set up the integral with y as the outer variable, note that y runs from 0 to 4, and that, at each fixed y, x runs from 0 to $\sqrt{4-y}$ (we use the positive square-root since x is positive here).



Therefore, we have

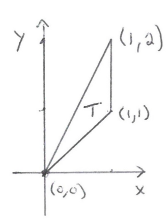
$$\iint_{\Omega} xy\, dA(x,y) = \int_0^4 \int_0^{\sqrt{4-y}} xy\, dxdy = \int_0^4 \frac{1}{2}y(4-y)dy = \int_0^4 2y - \frac{1}{2}y^2 dy = y^2 - \frac{1}{6}y^3\Big|_0^4 = 16 - \frac{1}{6}64 = \frac{16}{3}.$$

Definition 10. Let $\Omega \subset \mathbb{R}^n$ be a bounded set such that $\partial\Omega$ has measure zero. We define the **n-volume** of Ω to be

$$\mathrm{Vol}_n(\Omega) \overset{def}{=} \int_{\Omega} 1\, dV_n.$$

Even simple examples like computing the area of a region can illustrate some of the techniques one uses to study iterated integrals.

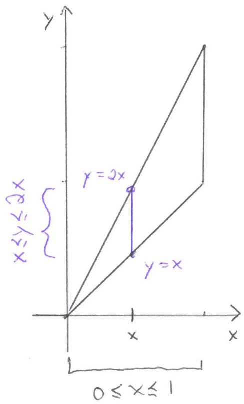
Example 24. Let T be the triangle with vertices at $(0,0), (1,1)$, and $(1,2)$. Compute the area of T by evaluating the double integral of $f(x,y) = 1$ over T. Set up the integral in two ways: first in the order $dydx$, and then in the order $dxdy$.



We first set up the iterated integrals with x as the outer variable. In this case, x runs from 0 to 1, so our double integral has the form

$$\iint_T 1\, dA = \int_0^1 \Big[\int_{\text{lower bound for y at x}}^{\text{upper bound for y at x}} 1\, dy\Big] dx.$$

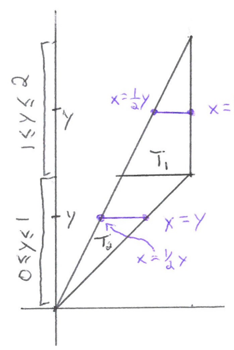
For each fixed $x \in [0,1]$, y runs from x to $2x$.



Hence,

$$\text{Area of T} = \iint_T 1\, dA = \int_0^1 \int_x^{2x} 1\, dy dx = \int_0^1 x dx = \frac{1}{2}.$$

To set up the iterated integrals with y as the outer integral, note first that y runs from 0 to 2. Computing the bounds for x at a given y is slightly complicated by the fact that the curves which bound x change at $y = 1$. We'll therefore split the region T into two pieces:



In the subregion T_1, i.e. when $1 \leq y \leq 2$, x runs from $\frac{1}{2}y$ to 1. In the subregion T_2, i.e. when $0 \leq y \leq 1$, x runs from $\frac{1}{2}y$ to y. Therefore, we have

$$\iint_{T_1} 1\, dA = \int_1^2 \int_{\frac{1}{2}y}^{1} 1\, dx dy = \int_1^2 1 - \frac{1}{2}y\, dy = 1 - \frac{4}{4} + \frac{1}{4} = \frac{1}{4},$$

and

$$\iint_{T_2} 1\, dA = \int_0^1 \int_{\frac{1}{2}y}^{y} 1\, dx dy = \int_0^1 \frac{1}{2}y\, dy = \frac{1}{4}.$$

So,

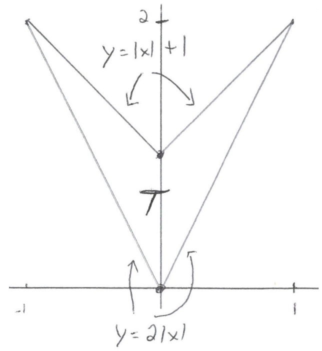
$$\iint_T 1\, dA = \iint_{T_1} 1\, dA + \iint_{T_2} 1\, dA = \frac{1}{2}.$$

Remark 11. In terms of shadows or projections, note that one can think of the double integral over T_2 from the previous example as

$$\iint_{T_2} f(x,y) dA = \int_{Shadow\ [0,1]} \int_{[\frac{1}{2}y, y]} f(x,y) dx dy.$$

This might seem like overkill here, but the analogous observation in three or more dimensions will be extremely helpful!

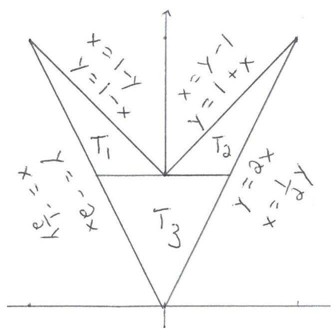
Example 25. Let T be the region bounded between the curves $y = |x| + 1$ and $y = 2|x|$. Set up the double integral of a general function $f(x, y)$ over T with (a) x as the outer variable, and (b) y as the outer variable.



We start with (a). Here, note that x runs from -1 to 1, while, for each fixed x, y runs from $2|x|$ to $|x| + 1$. Hence, we can write

$$\iint_T f(x, y)\, dA(x, y) \;=\; \int_{-1}^{1} \int_{2|x|}^{|x|+1} f(x, y)\, dydx.$$

Now, we consider (b). Note that, if we try to set up the integral where y runs from 0 to 2, then we run into trouble with the inner integral since, for y between 1 and 2, there are two separate intervals for x! To get around this, we can split the region T into three pieces as follows:



For T_1, we have

$$\iint_{T_1} f(x, y)\, dA(x, y) = \int_1^2 \int_{-\frac{1}{2}y}^{1-y} f(x, y)\, dxdy.$$

For T_2,

$$\iint_{T_2} f(x, y)\, dA(x, y) = \int_1^2 \int_{y-1}^{\frac{1}{2}y} f(x, y)\, dxdy.$$

Finally,

$$\iint_{T_3} f(x, y)\, dA(x, y) = \int_0^1 \int_{-\frac{1}{2}y}^{\frac{1}{2}y} f(x, y)\, dxdy.$$

So, in total,

$$\iint_T f(x, y)\, dA(x, y) = \int_1^2 \int_{-\frac{1}{2}y}^{1-y} f(x, y)\, dxdy + \int_1^2 \int_{y-1}^{\frac{1}{2}y} f(x, y)\, dxdy + \int_0^1 \int_{-\frac{1}{2}y}^{\frac{1}{2}y} f(x, y)\, dxdy.$$

You book gives names to the cases that we discussed above. Indeed, the definition given in your book of the 'type' of a region Ω in the plane is:

Type 1: Ω is **Type 1** if it can be written as $\Omega = \{(x, y) \mid a \le x \le b,\ \gamma(x) \le y \le \delta(x)\}$,

Type 2: Ω is **Type 2** if it can be written as $\Omega = \{(x, y) \mid c \le y \le d,\ \alpha(y) \le x \le \beta(y)\}$,

Type 3: Ω is **Type 3** if it is both Type 1 and Type 2.

That is, Type 1 regions are those which can be realized as the region between two functions y of x, while Type 2 regions are those which can be realized as the region between two functions x of y. Type 1 regions can be set up with y as the inner variable (without splitting up the region), while Type 2 regions can be set up with x as the inner variable (without splitting up the region). The last example is one which your book refers to as Type 1, and is NOT what the book refers to as Type 2.

This notion of 'Type' is not worth memorizing. You should, with a bit of work, be able to find the best way to set up the double integral over any region that you are given. The more examples you do, the more you will begin to easily recognize viable options for setting up an iterated integral. Indeed, we didn't need to resort to this 'type' nonsense in order to compute the previous examples. What is important is that each of these is an example of an *elementary* region, which is a form convenient for expressing multiple integrals over the region using iterated integrals.

Definition 11. Let $\Omega \subset \mathbb{R}^n$. We say that Ω is **elementary** if Ω is bounded and it is possible to write (for some ordering $x_{i_1}, \ldots, x_{i_n}$ of the variables $x_1, \ldots, x_n$)

$$\Omega = \{(x_1, \ldots, x_n) \ : \ a \le x_{i_1} \le b,\ \alpha_1(x_{i_1}) \le x_{i_2} \le \beta_1(x_{i_1}),\ \alpha_2(x_{i_1}, x_{i_2}) \le x_{i_3} \le \beta_2(x_{i_1}, x_{i_2}), \\ \ldots,\ \alpha_{n-1}(x_{i_1}, \ldots, x_{i_{n-1}}) \le x_{i_n} \le \beta_{n-1}(x_{i_1}, \ldots, x_{i_{n-1}})\},$$

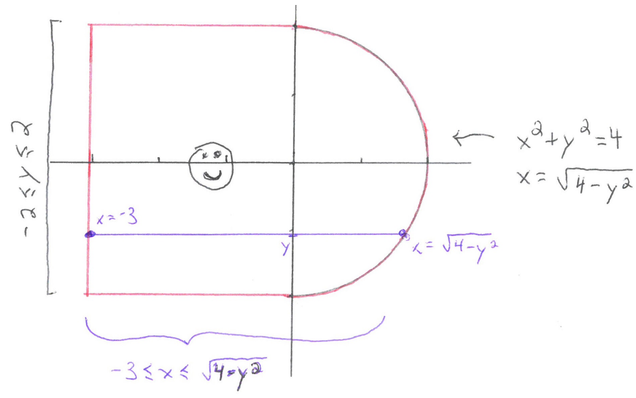
where $a, b \in \mathbb{R}$, and each function $\alpha_j, \beta_j : \mathbb{R}^j \to \mathbb{R}$ is continuous.

The integration region in each example we'll see is either elementary, or we will be able to split it up into a finite number of elementary regions. You should think of an elementary region as one which is "cut out" by the surfaces in $\mathbb{R}^n$ defined by the equations $x_{i_1} = a$, $x_{i_1} = b$, $x_{i_2} = \alpha_1(x_{i_1})$, $x_{i_2} = \beta_1(x_{i_1})$, and so on.

Remark 12. Elementary regions are useful largely because they have "nice" boundaries. For example, it is possible to prove that if $\Omega \subset \mathbb{R}^n$ is elementary, then $\partial\Omega$ has measure zero and therefore we can always integrate a continuous function over an elementary region. We will see more contexts in which elementary regions are useful.

Example 26. Sketch the region ☺ $\subset \mathbb{R}^2$ which is bounded on the right by the right-semicircle of radius 2 centered at $(0, 0)$, below by the line $y = -2$, above by the line $y = 2$, and on the left by the line $x = -3$. For a general function $f(x, y)$, express $\iint_{☺} f\, dA$ as an iterated integral in two different ways.

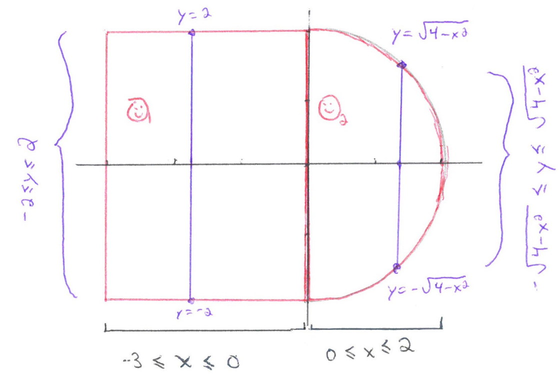
There are, as usual, two ways to set this up. The first is to use y as the outer variable. In this case, y runs from -2 to 2, and x runs from -3 to $\sqrt{4 - y^2}$.



Therefore, we have

$$\iint_{\smiley} f\,dA = \int_{-2}^{2}\int_{-3}^{\sqrt{4-y^2}} f(x,y)\,dxdy.$$

Now, if we want to set up the integral with x as the outer variable, then we need to split the region into two parts:



On $\smiley_1$, we have

$$\iint_{\smiley_1} f\,dA = \int_{-3}^{0}\int_{-2}^{2} f(x,y)\,dydx.$$

On $\smiley_2$, x runs from 0 to 2 and y runs from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$, so that

$$\iint_{\smiley_2} f\,dA = \int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dydx.$$

So,

$$\iint_{\smiley} f\,dA = \int_{-3}^{0}\int_{-2}^{2} f(x,y)\,dydx + \int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dydx.$$

Example 27. For $R > 0$, let $\Omega_R \subset \mathbb{R}^2$ denote the triangular region in the first quadrant bounded by the lines $y = 0$, $x = R$, and $x = y$. Show that

$$\frac{R^2}{2\pi} \le \iint_{\Omega_R} \sin\left(\frac{y}{x}\right) dA \le \frac{R^2}{4}.$$

(Suggestion: You may wish to first show that $\frac{2t}{\pi} \leq \sin(t) \leq t$ for $t \in \left[0, \frac{\pi}{2}\right]$.)

Here is a place where the integrand $\sin(\frac{y}{x})$ is actually undefined at a point in Ω_R (i.e. at $(0,0)$), and so we will by convention assume that we are actually studying $\iint\limits_{\Omega_R} f(x,y)\,dA(x,y)$, where

$$f(x,y) \overset{def}{=} \begin{cases} \sin\left(\frac{x}{y}\right) & \text{if } (x,y) \in \Omega_R,\ (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

To follow the suggestion, note that $\sin(t)$ is concave down on $[0,\pi]$, and therefore on $[0, \frac{\pi}{2}]$ its graph lies above the secant line passing through $(0,0)$ and $(\frac{\pi}{2}, 1)$. In other words, $\frac{2t}{\pi} \leq \sin(t)$ for $t \in [0, \frac{\pi}{2}]$.

For the other inequality, note that if $f(t) = t - \sin(t)$, then $f(0) = 0$ and $f'(t) = 1 - \cos(t) \geq 0$ for $t \geq 0$. Therefore, the Mean Value Theorem implies that $0 \leq \frac{f(t)-f(0)}{t-0}$ for all $t > 0$, so that $f(t) \geq 0$ for $t > 0$. But this exactly means that $t \geq \sin(t)$ for $t > 0$.

Now that we have the double inequality in hand, note that in the region Ω_R we have $0 \leq \frac{y}{x} \leq 1$, and therefore $\frac{2y}{\pi x} \leq \sin\left(\frac{y}{x}\right) \leq \frac{y}{x}$ throughout Ω_R (and that $\sin(\frac{y}{x})$ is bounded on Ω_R, an. We therefore use monotonicity of the integral to write

$$\frac{2}{\pi} \iint\limits_{\Omega_R} \frac{y}{x}\,dA(x,y) \leq \iint\limits_{\Omega_R} \sin\left(\frac{y}{x}\right) dA(x,y) \leq \iint\limits_{\Omega_R} \frac{y}{x}\,dA(x,y).$$

We therefore need only compute $\iint_{\Omega_R} \frac{y}{x} dA$. To do this, we use Fubini's theorem to write

$$\iint\limits_{\Omega_R} \frac{y}{x}\,dA(x,y) = \int_0^R \int_y^R \frac{y}{x}\,dxdy = \int_0^R y(\ln(R) - \ln(y))dy.$$

We would certainly compute this integral, but this seems like a lot of work. Indeed, the integral is improper since the integrand is undefined when $y = 0$, so we must actually compute (using integration by parts and l'Hopital's rule)

$$\begin{aligned} \lim_{\epsilon \to 0+} \int_\epsilon^R y(\ln(R) - \ln(y))dy &= \lim_{\epsilon \to 0+} \frac{R^2}{2}(\ln(R) - \ln(R)) - \frac{\epsilon^2}{2}(\ln(R) - \ln(\epsilon)) + \frac{1}{2}\int_0^R \frac{y}{2}dy \\ &= \lim_{\epsilon \to 0+} \frac{R^2}{4} - \frac{\epsilon^2}{4} = \frac{R^2}{4}. \end{aligned}$$

If we had just set up the integral in the other order, we would have had a much easier time:

$$\iint\limits_{\Omega_R} \frac{y}{x} dA = \int_0^R \int_0^x \frac{y}{x} dydx = \int_0^R \frac{x}{2} dx = \frac{R^2}{4}.$$

At any rate, plugging this value in for the integral above gives us

$$\frac{R^2}{2\pi} \leq \iint\limits_{\Omega_R} \sin\left(\frac{y}{x}\right) dA(x,y) \leq \frac{R^2}{4},$$

as desired.

Example 28. Evaluate $☺ = \displaystyle\int_0^{2\pi}\int_0^1 \sin(y)\sqrt{1-x^2}dxdy$.

We can evaluate the inner integral via trig substitution. Let $x = \sin(\theta)$, so that $dx = \cos(\theta)d\theta$. Then

$$\begin{aligned} ☺ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin(y)\sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta dy \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin(y)\cos^2(\theta)d\theta dy \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin(y)\frac{1+\cos(2\theta)}{2}d\theta dy \\ &= \int_0^{2\pi} \frac{1}{4}\sin(y)(2\theta+\sin(2\theta))\Big|_0^{\frac{\pi}{2}} dy \\ &= \int_0^{2\pi} \frac{\pi}{4}\sin(y)dy \\ &= 0. \end{aligned}$$

This was a lot of work. However, if we changed the order of integration we would get

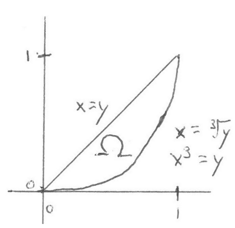
$$\begin{aligned} ☺ &= \int_0^1\int_0^{2\pi} \sin(y)\sqrt{1-x^2}dydx \\ &= \int_0^1 0dx \\ &= 0. \end{aligned}$$

This example illustrates an important point: changing the order of integration can sometimes make a difficult (or impossible!) integral easy to evaluate via the Fundamental Theorem of Calculus.

Why might an integral be impossible to evaluate using the Fundamental Theorem of Calculus? It turns out (according to a theorem of Liouville in the early 19th century) that some continuous functions, such as e^{x^2} or $\frac{\sin(x)}{x}$ (where this last function is defined to be 1 when $x = 0$), do not have antiderivatives which can be written down explicitly in terms of the 'nice' functions that we normally work with (i.e. rational functions and polynomials, roots, trigonometric functions, exponential functions, etc.). Contrast this with the Fundamental Theorem of Calculus, which says that $F(x) = \int_0^x \frac{\sin(t)}{t}dt$ is an antiderivative of $\frac{\sin(x)}{x}$.

Example 29. Evaluate $☺ = \displaystyle\int_0^1\int_y^{\sqrt[3]{y}} e^{x^2}\,dxdy$.

Right away, we see that the inner integral is impossible to evaluate explicitly using the Fundamental Theorem of Calculus. However, we can try to change the order of integration to see if something happens. First, we need to see this iterated integral as a double integral over some region Ω. Note that Ω is given by $0 \le y \le 1$ and $y \le x \le \sqrt[3]{y}$. This region is sketched below:



We can change the order of integration by writing

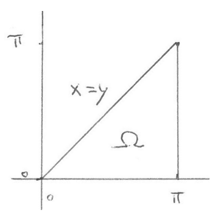
$$\begin{aligned} ☺ &= \int_0^1 \int_{x^3}^{x} e^{x^2}\,dydx \\ &= \int_0^1 (x - x^3)e^{x^2}\,dx. \end{aligned}$$

We can compute this by writing $x = \sqrt{u}$, so that $dx = \frac{1}{2\sqrt{u}}du$. Making this substitution,

$$☺ = \int_0^1 \frac{1}{2}(1-u)e^u du = \frac{1}{2}(1-u)e^u + \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}e - 1.$$

Example 30. Compute $I = \displaystyle\int_0^{\pi}\int_y^{\pi} \frac{\sin(x)}{x}dxdy.$

Once again, we see that the integrand doesn't have an antiderivative which can be written down nicely. However, we can change the order of integration (and cross our fingers!) to try to make the integral possible to evaluate. First we sketch the region of integration Ω below:

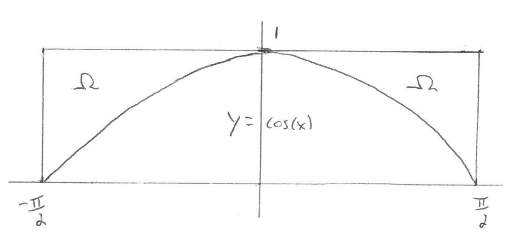


We can therefore write

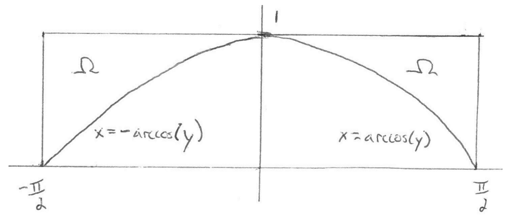
$$\begin{aligned} I &= \int_0^{\pi}\int_0^{x} \frac{\sin(x)}{x}dydx \\ &= \int_0^{\pi} \sin(x)dx \\ &= 2. \end{aligned}$$

Example 31. Compute $☺ = \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\cos(x)}^{1} \frac{y}{\frac{\pi}{2} - \arccos(y)}dydx.$

The integrand looks horrible enough that it may not be possible to evaluate using the Fundamental Theorem of Calculus. Once again, we need to reverse the order of integration in order to (try to!) compute the double integral. Here, the region of integration is a bit more complicated than in the previous examples:



We have to be careful here, since working with inverse trigonometric functions can be tricky. We carefully label the edges of the domain below:



We can therefore write

$$\begin{aligned}
☺ &= \int_0^1 \int_{-\frac{\pi}{2}}^{-\arccos(y)} \frac{y}{\frac{\pi}{2} - \arccos(y)} dxdy + \int_0^1 \int_{\arccos(y)}^{\frac{\pi}{2}} \frac{y}{\frac{\pi}{2} - \arccos(y)} dxdy \\
&= \int_0^1 ydy + \int_0^1 ydy \\
&= 1.
\end{aligned}$$

Lecture 9: Triple Integrals

Learning Objectives:

- Evaluate a triple integral by setting up an appropriate iterated integral.
- Change the order or integration in an iterated integral.

Example 32. For a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$, sketch the region $E \subset \mathbb{R}^3$ satisfying

$$\int_0^2 \int_0^3 \int_0^{5-x-y} f(x,y,z)\, dzdxdy = \iiint_E f(x,y,z)\, dV,$$

and write $\iiint_E f(x,y,z)\, dV$ as an iterated integral in the order $dxdzdy$.

Reconstructing the region of integration from the iterated integrals is very similar to the two-dimensional case.

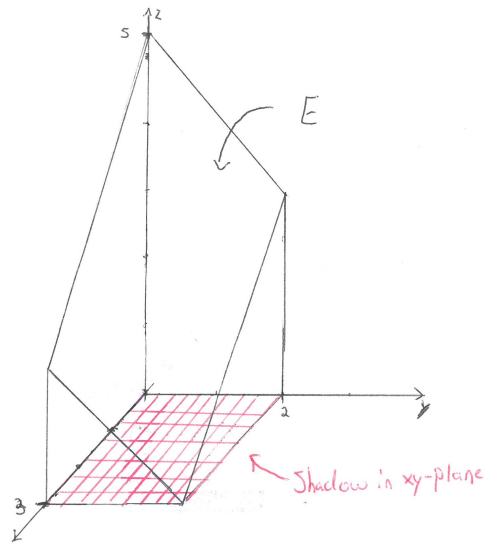
We can think of the bounds on the integrals as giving up the surfaces which bound the region over which we are integrating. For example, the bounds in this integral suggest that the region E is bounded by:

- The planes $y = 0$ and $y = 2$,
- The planes $x = 0$ and $x = 3$,
- The planes $z = 0$ and $z = 5 - x - y$.

The fact that the (outer) and (middle) integrals have the form $\int_0^2 \int_0^3 \cdots dxdy$ says that the **shadow** (or orthogonal projection) of E onto the xy-plane is exactly the rectangle $[0,3] \times [0,2]$. Indeed, we might therefore write

$$\int_0^2 \int_0^3 \int_0^{5-x-y} f(x,y,z)\, dzdxdy = \iint_{[0,3]\times[0,2]} \int_0^{5-x-y} f(x,y,z)\, dz\, dA(x,y).$$

Once we fix x and y in this shadow, then the inner integral $\int_0^{5-x-y} f(x,y,z)dz$ tells us that we should integrate the function $f(x,y,z)$ from $z = 0$ to $z = 5 - x - y$. This means that we are integrating over the region bounded between the xy-plane and the plane $x + y + z = 5$, above the rectangle $[0,3] \times [0,2]$ (in the xy-plane). We can sketch the region E as follows:

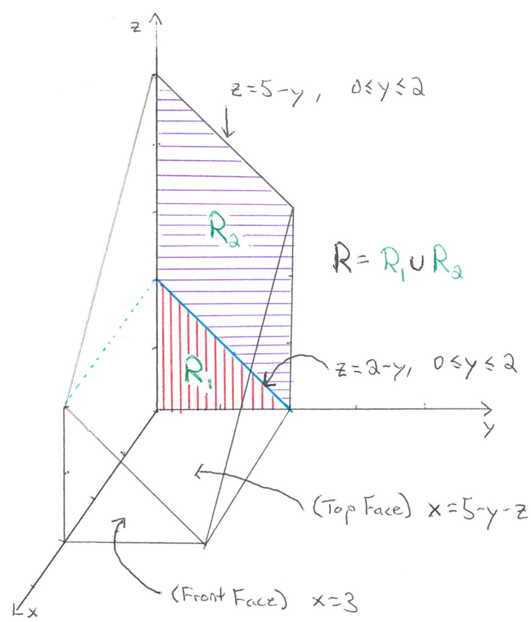


We can get an idea for the curves that form the edges of E by noting that, for example, in the xz-plane the upper edge of E is the intersection of $x+y+z=5$ and $y=0$, or rather the line $z=5-x$ for $0 \le x \le 3$. Similar considerations give the other three edges on the top of E are

$$\begin{aligned}\text{In the } yz-\text{plane}: &\quad z=5-y, \quad 0 \le y \le 2,\\ \text{In the plane } x=3: &\quad z=2-y, \quad 0 \le y \le 2,\\ \text{In the plane } y=2: &\quad z=3-x, \quad 0 \le x \le 3.\end{aligned}$$

We now want to write $\iiint_E f(x,y,z)dV$ as an iterated integral in the order $dxdzdy$. Our first task is to compute the shadow of E in the zy-plane. The shadow is the region R where $0 \le y \le 2$ and $0 \le z \le 5-y$ (see below).

Our next task is to compute the bounds for the inner integral. However, we run into a small problem here because although the lower bound for x is always 0, the upper bound changes depending on where y and z are located in the shadow. Let's split the shadow R into two regions R_1 and R_2, where in R_1 the upper bound for x is 3, and in R_2 the upper bound for x is $x=5-y-z$.



The dividing line between the regions R_1 and R_2 is the shadow of the intersection of the plane $x+y+z=5$ and the plane $x=3$. In other words, the regions R_1 and R_2 are divided by the line $z=2-y$ in the zy-plane.

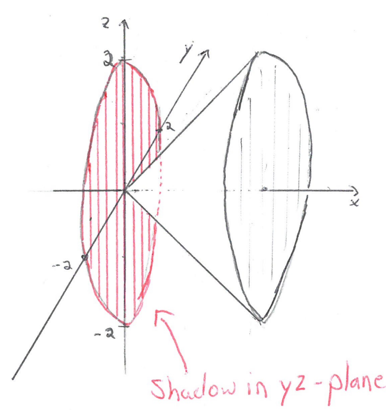
We therefore set up the iterated integrals as follows:

$$\begin{aligned}\iiint_E f(x,y,z)dV &= \iint_{R_1}\int_0^3 f(x,y,z)dxdA(z,y)+\iint_{R_2}\int_0^{5-y-z} f(x,y,z)dxdA(z,y)\\ &= \int_0^2\int_0^{2-y}\int_0^3 f(x,y,z)dxdzdy+\int_0^2\int_{2-y}^{5-y}\int_0^{5-y-z} f(x,y,z)dxdzdy.\end{aligned}$$

Remark 13. This example also illustrates an important point, which generalizes from double integrals: the bounds for the inner integral can depend on (middle) and (outer), the bounds for the middle integral can only depend on (outer), and the bounds for the outer integral must be constants!

Remark 14. Note in the last example that we found the boundary of the shadow by computing the shadow of the intersection of two surfaces. This type of geometric reasoning will be very helpful going forward.

Example 33. Let E be the region bounded by the cone $x=\sqrt{y^2+z^2}$ and the plane $x=2$. Set up the iterated integral of a general function $f(x,y,z)$ over this region in two different ways.

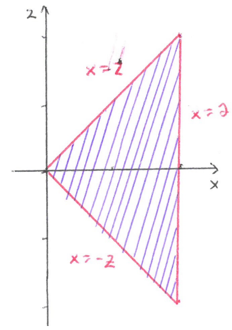


Here, it seems to make the most sense to use x as the (inner) variable, since we can easily write $\sqrt{y^2+z^2}\le x\le 2$, which will give us the bounds for the inner integral.

For the middle and outer integrals, we need to know what the shadow of E is in the yz-plane. For this region, it is not hard to see that the shadow of E is the (filled in) disc $y^2+z^2\le 4$ of radius 2 centered at $(0,0)$ (in the yz-plane). Choosing z to be the (outer) variable, we have $-2\le z\le 2$, and $-\sqrt{4-z^2}\le y\le\sqrt{4-z^2}$. Therefore,

$$\iiint_E f dV=\int_{-2}^2\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{\sqrt{y^2+z^2}}^2 f(x,y,z)dxdydz.$$

Let's now set up the iterated integral of a general function $f(x,y,z)$ over this region in the order $dydzdx$. The shadow of this region in the xz-plane is the triangle bounded by the lines $x=2$, $z=x$, and $z=-x$.



Therefore, we have $0 \leq x \leq 2$ and $-x \leq z \leq x$.

Since the cone is described by $x^2 - z^2 = y^2$, we have $-\sqrt{x^2 - z^2} \leq y \leq \sqrt{x^2 - z^2}$ for the bounds for the inner integral.

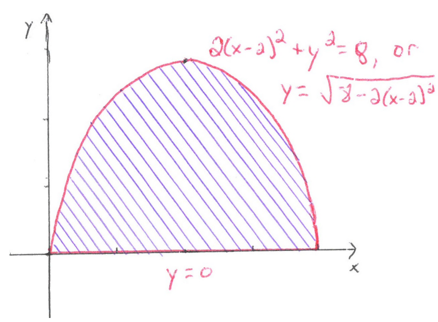
Therefore,

$$\iiint_E f dV = \int_0^2 \int_{-x}^{x} \int_{-\sqrt{x^2-z^2}}^{\sqrt{x^2-z^2}} f(x, y, z) dy dz dx.$$

Example 34. Sketch or describe the region E of integration of

$$\int_0^4 \int_0^{\sqrt{8-2(x-2)^2}} \int_{4-x}^{\sqrt{16-x^2-y^2}} f(x, y, z) dz dy dx.$$

We first describe the shadow of E in the xy-plane. This is the region between $x = 0$ and $x = 4$ which is bounded below by $y = 0$ and above by the curve $y = \sqrt{8 - 2(x-2)^2}$, which is the upper half of the ellipse $2(x-2)^2 + y^2 = 8$.

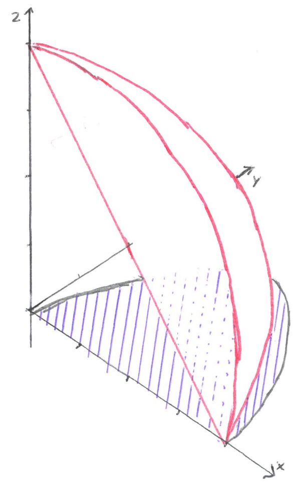


The actual region sits above this shadow between the plane $z = 4-x$ and the surface $z = \sqrt{16 - x^2 - y^2}$, which is the upper-half of the sphere of radius 4 centered at the origin.

Note that when $y = \sqrt{8 - 2(x-2)^2}$, we have

$$\sqrt{16 - x^2 - y^2} = \sqrt{16 - x^2 - (8 - 2(x-2)^2)} = \sqrt{x^2 - 8x + 16} = |4 - x| = 4 - x$$

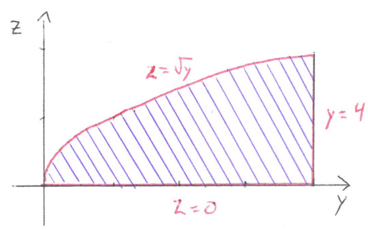
(since $x \leq 4$), so the plane and sphere intersect above the curve $y = \sqrt{8 - 2(x-2)^2}$ in the xy-plane. Therefore, this region is the 'half lens' that you get by slicing the (solid) sphere with the plane $z = 4-x$ and the xz-plane.



Example 35. Sketch or describe the region E of integration of

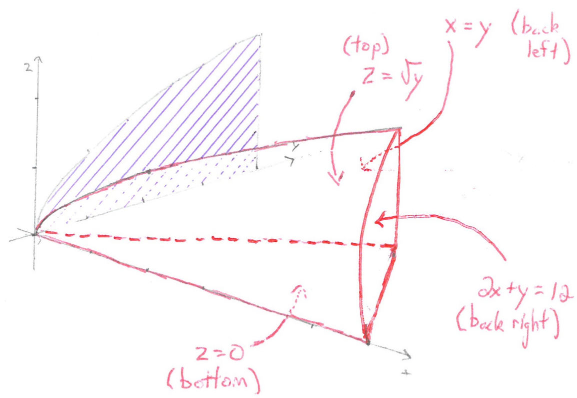
$$\int_0^4 \int_0^{\sqrt{y}} \int_y^{6-\frac{1}{2}y} f(x,y,z)\,dx\,dz\,dy.$$

The shadow of this region lies in the yz-plane, as pictured



Now we need to determine what E looks like. The boundary of E is formed by the surfaces $z=0$ (on the bottom), $z=\sqrt{y}$ (part of the front/top), $x=y$ (the left/back), $2x+y=12$ (the right/back face).

The faces $x=y$ and $2x+y=12$ intersect at the vertical line $x=4, y=4$. The top edge of E on the face $x=y$ is the intersection of $z=\sqrt{y}$ and $x=y$, while the top edge of E on the face $2x+y=12$ is the intersection of the plane $2x+y=12$ with $z=\sqrt{y}$. Therefore, we can sketch E as follows:

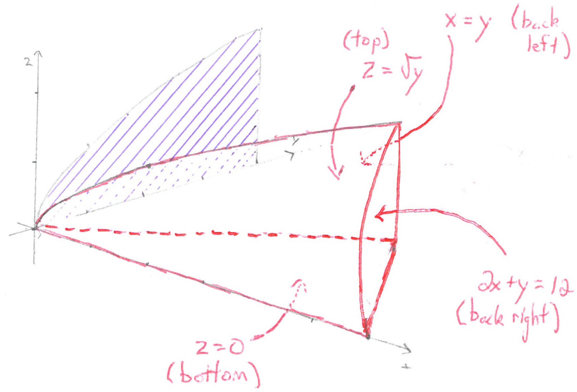


Example 36. Consider the iterated integral

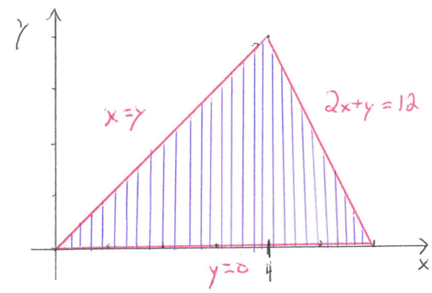
$$I = \int_0^4 \int_0^{\sqrt{y}} \int_y^{6-\frac{1}{2}y} f(x,y,z)\,dx\,dz\,dy.$$

Rewrite the iterated integral in the order $dzdydx$ and $dydzdx$.

Recall the sketch that we made of the region of integration E:



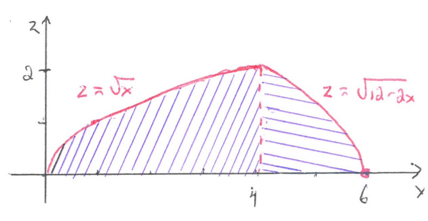
To write the integral with z as the inner variable, we first need to know what is the shadow of E in the xy-plane. This shadow is the triangle T bounded by the lines $y = 0$, $x = y$, and $2x + y = 12$:



At each point over the shadow, we have $0 \leq z \leq \sqrt{y}$, which gives the bound for the inner integral. For the outer double integral, though, we need to split the region into two pieces. That is, for $0 \leq x \leq 4$ we have $0 \leq y \leq x$, and for $4 \leq x \leq 6$ we have $0 \leq y \leq 12 - 2x$. Therefore, we have

$$I = \int_0^4 \int_0^x \int_0^{\sqrt{y}} f(x, y, z) dzdydx + \int_4^6 \int_0^{12-2x} \int_0^{\sqrt{y}} f(x, y, z) dzdydx.$$

For the order $dydxdz$, we first need to know what is the shadow of the integration region in the xz-plane. The shadow lives above the x-axis in the interval $0 \leq x \leq 6$. For x between 0 and 4 we see that the upper edge of the shadow is given by $z = \sqrt{y} = \sqrt{x}$, while for x between 0 and 4 we have that the upper-edge of the shadow is given by $z = \sqrt{y} = \sqrt{12 - 2x}$. The shadow is sketched below:



Now, the formula that we use to find the upper-bound for y depends on where in the shadow we look. If $0 \leq x \leq 4$, then we have $z^2 \leq y \leq x$, while if $4 \leq x \leq 6$ we have $z^2 \leq y \leq 12 - 2x$. Therefore, we need to split the outer double integral as follows:

$$I = \int_0^4 \int_0^{\sqrt{x}} \int_{z^2}^{x} f(x, y, z) dydzdx + \int_4^6 \int_0^{\sqrt{12-2x}} \int_{z^2}^{12-2x} f(x, y, z) dydzdx.$$

This last example illustrates that, just as for iterated integrals for two-variable functions, it may be necessary to split up the domain when changing the order of integration for an iterated integral of a three-variable function. In the next example, we see that sometimes rather messy sums of integrals can be rewritten in a much nicer way by changing the order of integration.

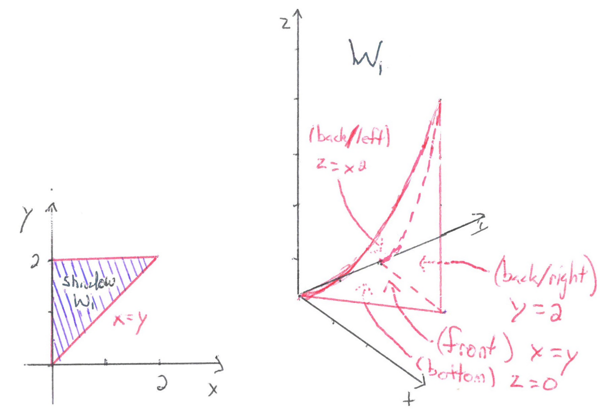
Example 37. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Write the sum of the triple integrals

$$☺ = \int_0^2 \int_0^y \int_0^{x^2} f(x,y,z)dzdxdy + \int_0^4 \int_0^{\sqrt{z}} \int_{\sqrt{z}}^2 f(x,y,z)dydxdz$$

as an iterated integral with respect to the order $dzdydx$.

Let I denote the first iterated integral, and II denote the second. Let's rewrite both I and II in the order specified, and then see how they relate to one another.

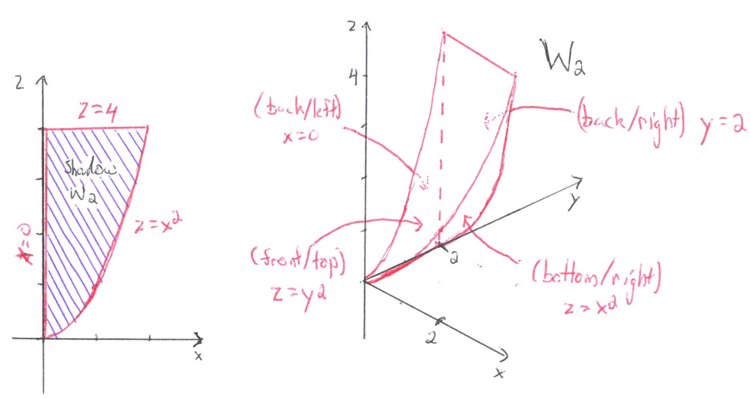
For I, we note that the region of integration W_1 lies between the surfaces $z = 0$ (i.e. the xy-plane) and the parabolic cylinder $z = x^2$. The shadow of W_1 in the xy-plane is the triangle bounded by the lines $x = 0$, $y = 2$, and $x = y$, so we can sketch the shadow of W_1 and W_1 as follows:



Now, to rewrite this in the order $dzdydx$ we only need to switch the order of integration for the outer and middle variables. That is, we now have $0 \le x \le 2$ and $x \le y \le 2$, so that

$$I = \int_0^2 \int_x^2 \int_0^{x^2} f(x,y,z)dzdydx.$$

We turn our attention to II and its region of integration W_2. First, note that W_2 is bounded between the (parabolic cylinder) $y = \sqrt{z}$ (or $z = y^2$) and the plane $y = 2$. The shadow of W_2 in the xz-plane is region bounded by the lines $x = 0$, $z = 4$, and the curve $z = x^2$. We sketch W_2 below:



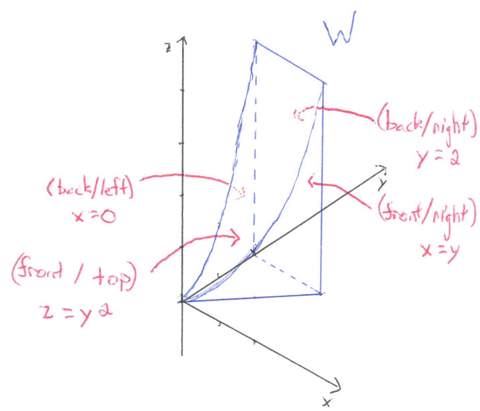
Our first step in changing the order of integration is to note that the shadow of W_2 in the xy-plane is bounded by the lines $x = 0$, $y = 2$, and $x = \sqrt{z} = y$. (Note that this is the same shadow that we got for W_1!) Above a fixed x and y, z runs from x^2 (the surface $z = x^2$ bounds the region below) to y^2 (the surface $z = y^2$ bounds W_2 from above). Therefore, we have

$$II = \int_0^2 \int_x^2 \int_{x^2}^{y^2} f(x, y, z) dz dy dx.$$

Note that both I and II have the same shadow in the xy-plane, and that therefore

$$☺ = I + II = \int_0^2 \int_x^2 \left[\int_0^{x^2} f(x, y, z) dz + \int_{x^2}^{y^2} f(x, y, z) dz \right] dy dx = \int_0^2 \int_x^2 \int_0^{y^2} f(x, y, z) dz dy dx.$$

The region of integration W of this integral is sketched below:



Lecture 10: More Integrals

Learning Objectives:

- Compute multiple integrals over high-dimensional sets using slicing or shadows.

Today we look at a helpful alternate approach to the analysis techniques from the past few days relating iterated integrals to multiple integrals.

Shadows vs. Slices

In the past few lectures we focused on how to analyze double and triple integrals using iterated integrals, and this analysis involved a significant amount of sketching and geometric reasoning. It also involved moving fluently between multiple integrals and iterated integrals. In particular, we noted that if the value of some (say) triple integral $\iiint_E f(x,y,z)\,dV$ can be expressed as an iterated integral such as

$$\int_0^2 \int_0^y \int_0^{x^2} f(x,y,z)\,dz\,dx\,dy,$$

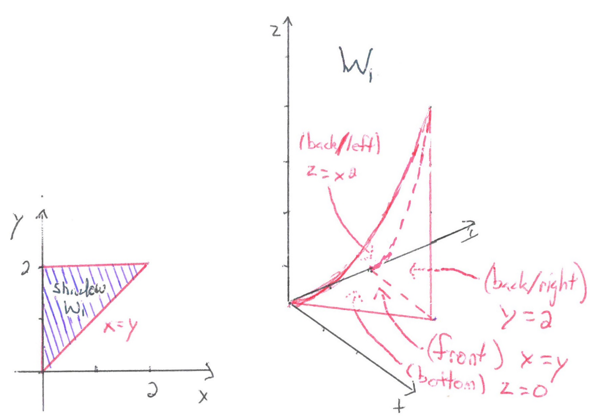
then we might attempt to recover E by analyzing the integration bounds on the iterated integrals. In particular, we treated the inner-most integral as a function of the outer two variables

$$☺(x,y) \overset{def}{=} \int_0^{x^2} f(x,y,z)\,dz,$$

and then considered the resulting expression as an iterated integral representing a double integral:

$$\begin{aligned}\int_0^2 \int_0^y \int_0^{x^2} f(x,y,z)\,dz\,dx\,dy &= \int_0^2 \int_0^y ☺(x,y)\,dx\,dy \\ &= \iint_S ☺(x,y)\,dA(x,y) \\ &= \iint_S \int_0^{x^2} f(x,y,z)\,dz\,dA(x,y),\end{aligned}$$

where S was the shadow of E in the coordinate plane corresponding to the two "outer variables". In this case, S is the triangular region described by $0 \le y \le 2$ and $0 \le x \le y$ in the xy-plane. This allowed us to view E as the region between the graphs of $z = 0$ and $z = x^2$ as (x,y) range through S.



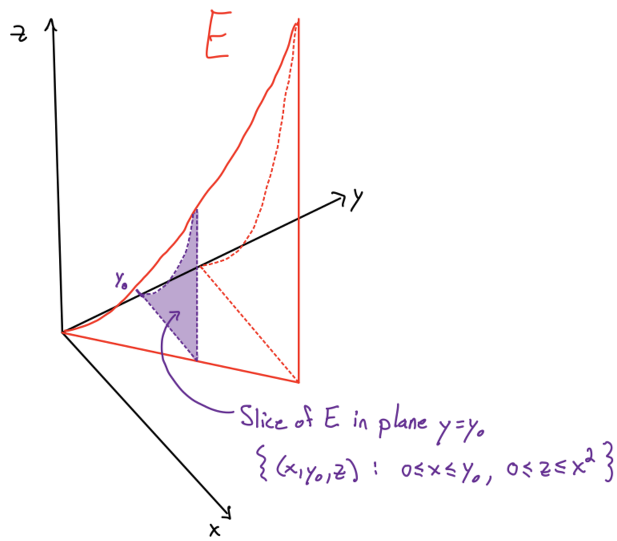
One powerful observation is that the reasoning used above (i.e. 'grouping' some of the single-variable integrations in an iterated integral as a multiple integration) can be applied in other ways, with different geometric interpretations. For example, we can instead group the "inner" single-variable integrations:

$$\int_0^2 \int_0^y \int_0^{x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^2 \left[\int_0^y \int_0^{x^2} f(x,y,z)\,dz\,dx \right] dy = \int_0^2 \iint\limits_{\substack{0\le x\le y\\ 0\le z\le x^2}} f(x,y,z)\,dA(z,x)\,dy.$$

For each fixed y_0 between 0 and 2, the inequalities $0 \le x \le y_0$ and $0 \le z \le x^2$ characterize the values of x and z such that $(x, y_0, z) \in E$. This set

$$\{(x,y,z) \in E \ : \ y = y_0\} = \{(x, y_0, z) \ : \ 0 \le x \le y, \ 0 \le z \le x^2\}$$

is called the **slice** of E in the plane $y = y_0$. One can define the slices of E with respect to other planes as well. In higher dimensions, we talk about the slice of a set in a hyperplane.

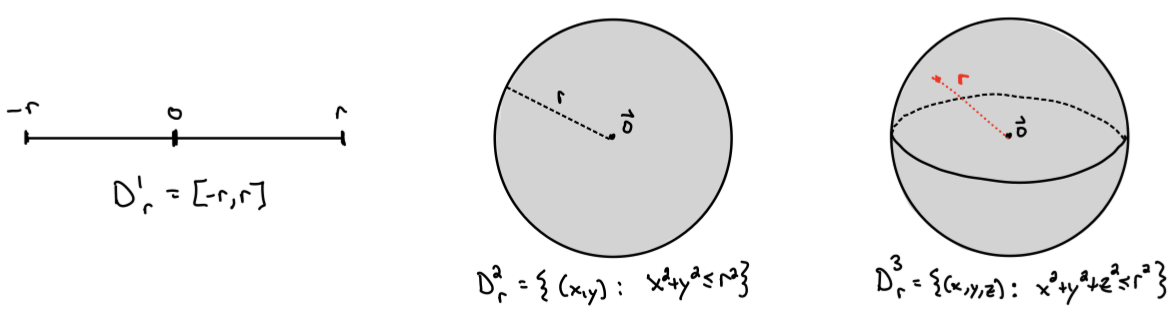


This technique of slicing is not so useful for some applications (like explicit computations of a multiple integral of an interesting function), as we usually need the inner-most integral to be a single-variable integral in order to get a computation off the ground. It *is* useful, however, when we are trying to analyze certain high-dimensional problems like computing volumes of standard sets like balls.

Example 38. For each $n \ge 1$ and $r > 0$, let

$$D_r^n = \{\vec{x} \in \mathbb{R}^n \ : \ \|\vec{x}\| \le r\}$$

denote the closed n-dimensional ball of radius r centered at $\vec{0}$. Note that $D_r^n = B_r(\vec{0}) \cup S^{n-1}$, where $B_r(\vec{0})$ is the open ball of radius r centered at $\vec{0}$ in $\mathbb{R}^n$ and $S^{n-1} = \partial B_r(\vec{0})$ is the unit hypersphere in $\mathbb{R}^n$. D_r^1, D_r^2, and D_r^3 are pictured below.



One of the famous computations in mathematics (which you will do on your homework!) is to compute the n-volume of D^n_r for each $n \geq 1$. Your first step will be to show that $\mathrm{Vol}_n(D^n_r) = r^n \mathrm{Vol}_n(D^n_1)$ (using results we will discuss next time), so the rest of the problem boils down to computing $\mathrm{Vol}_n(D^n_1)$.

The first couple computations are not difficult. For example,

$$\mathrm{Vol}_1(D^1_1) = \mathrm{Vol}_1([-1, 1]) = (1 - (-1)) = 2$$

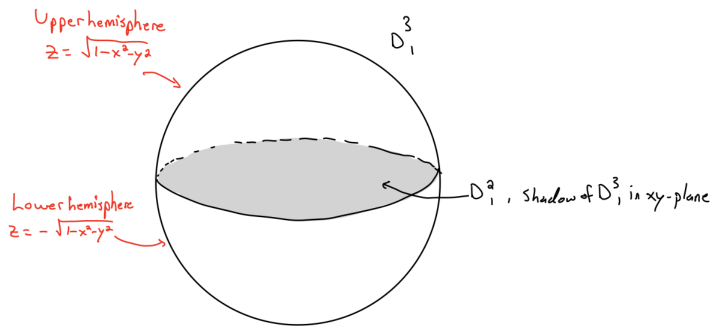
(so that $\mathrm{Vol}_1(D^1_r) = r\mathrm{Vol}_1(D^1_1) = 2r$) and (what is a standard problem involving the trigonometric substitution $x = \sin(t)$ in single-variable calculus, and using a power-reducing formula and the fact that $\cos(t) \geq 0$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$)

$$\begin{aligned}
\mathrm{Vol}_2(D^2_1) &= \iint_{D^2_1} 1\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dydx \\
&= \int_{-1}^{1} 2\sqrt{1-x^2}\, dx \\
&= \int_{-\pi/2}^{\pi/2} 2\sqrt{1-\sin^2(t)}\cos(t)\, dt \\
&= \int_{-\pi/2}^{\pi/2} 2\sqrt{\cos^2(t)}\cos(t)\, dt \\
&= \int_{-\pi/2}^{\pi/2} 2|\cos(t)|\cos(t)\, dt \\
&= \int_{-\pi/2}^{\pi/2} 2\cos^2(t)\, dt \\
&= \int_{-\pi/2}^{\pi/2} (\cos(2t) + 1)\, dt \\
&= \frac{\sin(2t)}{2} + t\Big]_{-\pi/2}^{\pi/2} \\
&= \pi,
\end{aligned}$$

so that $\mathrm{Vol}_2(D^2_r) = r^2\mathrm{Vol}_2(D^2_1) = \pi r^2$, which is the well-known formula for the area of a disc of radius r.

The computation for $\mathrm{Vol}_3(D^3_1)$ is a bit more involved. The shadow of D^3_1 in the xy-plane is the ball D^2_1, and for each choice of $(x, y) \in D^2_1$ have that $-\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}$, and therefore we can write

$$\begin{aligned}
\mathrm{Vol}_3(D^3_1) &= \iint_{D^2_1} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz\, dA(y, x) \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz\, dy\, dx.
\end{aligned}$$

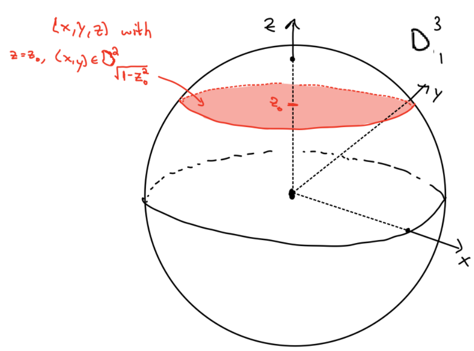


One can compute this iterated integral using reasoning similar to the first problem by making the substitution $y = \sqrt{1-x^2}\cos(t)$ to obtain

$$\begin{aligned}
\mathrm{Vol}_3(D_1^3) &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\,dy\,dx \\
&= \int_{-1}^{1}\int_{-\pi/2}^{\pi/2} 2\sqrt{(1-x^2)(1-\sin^2(t))}\sqrt{1-x^2}\cos(t)\,dt\,dx \\
&= \int_{-1}^{1}(1-x^2)\pi\,dx \\
&= \frac{4\pi}{3}.
\end{aligned}$$

This computation could have been much easier, though, with the observation that the slice of D_1^3 in the plane $z = z_0$ consists of points (x, y, z_0) satisfying $x^2 + y^2 + z_0^2 \le 1$, or rather $x^2 + y^2 \le 1 - z_0^2$, and therefore

$$\begin{aligned}
\mathrm{Vol}_3(D_1^3) &= \int_{-1}^{1}\iint_{D^2_{\sqrt{1-z^2}}} 1\,dA(x,y)\,dz \\
&= \int_{-1}^{1}\mathrm{Vol}_2(D^2_{\sqrt{1-z^2}})\,dz \\
&= \int_{-1}^{1}\pi(\sqrt{1-z^2})^2\,dz \\
&= \int_{-1}^{1}\pi(1-z^2)\,dz \\
&= \frac{4\pi}{3}.
\end{aligned}$$



The key idea here is that we were able to use the formula for $\mathrm{Vol}_2(D_r^2)$ to compute $\mathrm{Vol}_3(D_1^3)$, because the slices of the closed 3-dimensional ball D_1^3 in planes of the form $z = z_0$ are simply closed 2-dimensional balls! The same idea can be used to compute $\mathrm{Vol}_n(D_r^n)$ for every $n \geq 1$, and there are a couple ways to approach the argument. One of your homework problems walks you though a possible solution.

Lecture 11: Change of Variables

Learning Objectives:

- Determine when it is possible to change variables in a multiple integral.
- Change variables in a multiple integral.

Before we start our general discussion of change of variables in multiple integrals, let's look at a single-variable example to motivate the ideas. We'll use single-variable substitution freely in this example (which is usually proved in Calculus I using the Fundamental Theorem of Calculus and the chain rule).

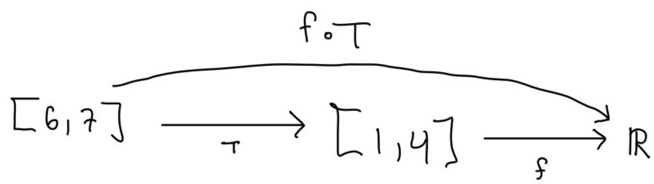
Example 39. Consider $f : [1,4] \to \mathbb{R}$, $f(x) = e^{8-\sqrt{x}}$. This function is continuous (and therefore integrable) on $[1,4]$, and the integral can be expressed as in your single-variable calculus course:

$$\int_{[1,4]} f(x)\, dV_1(x) = \int_1^4 e^{8-\sqrt{x}}\, dx.$$

If we wanted to compute this integral, the first step would be to make the substitution $u = 8 - \sqrt{x}$. Since $1 \le x \le 4$ and $8 - \sqrt{x}$ is a decreasing function of x, $6 \le u \le 7$. Moreover, if we solve this equation for x as $x = (8-u)^2$, then we see that $dx = 2(8-u)(-1)du$, so that

$$\int_{[1,4]} f(x)\, dV_1(x) = \int_1^4 e^{8-\sqrt{x}}\, dx = \int_7^6 e^u 2(8-u)(-1)\, du = \int_6^7 e^u 2(8-u)\, du = \int_{[6,7]} e^u 2(8-u)\, dV_1(u).$$

To view this result in a way that generalizes to multiple integrals, note that if $T : [6,7] \to [1,4]$ is $T(u) = (8-u)^2$, then when we make the subsitution $x = T(u)$ we are actually replacing $f(x) = e^{8-\sqrt{x}}$ (defined on $[1,4] = T([6,7])$) with $f(T(u)) = e^{8-\sqrt{(8-u)^2}} = e^u$ (a function defined on $[6,7]$).

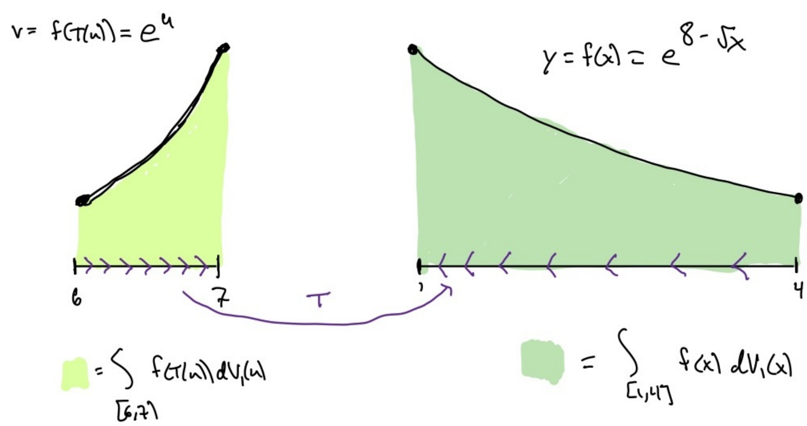


The function $f \circ T : [6,7] \to \mathbb{R}$ is called the **pullback** of f by T. (The picture above motivates the name, as composing f with T "pulls back" the domain of f from $[1,4] = T([6,7])$ to $[6,7]$.)[8] Note that $f(T(u))$ is integrable over $[6,7]$, and therefore we might expect that

$$\int_{T([6,7])} f(x)\, dV_1(x) \quad \text{and} \quad \int_{[6,7]} f(T(u))\, dV_1(u)$$

are related somehow.

[8] Indeed, the idea of "pulling back" an object in this way is an important idea that we will see return a few more times this quarter.



We don't necessarily expect equality to hold here because the map T stretches out the interval $[6,7]$ when mapping it onto $[1,4]$. This stretching would certainly prevent equality for "corresponding" Riemann sums of f and $f \circ T$ since if $c_i = T(d_i)$ and $[a_i, b_i] = T([\alpha_i, \beta_i])$ we expect $[a_i, b_i]$ and $[\alpha_i, \beta_i]$ to have different lengths, and therefore

$$f(c_i)|b_i - a_i| = f(T(d_i))|T(\beta_i) - T(\alpha_i)| \neq f(T(d_i))|\beta_i - \alpha_i|.$$

However, we can account for the change in lengths using the Mean Value Theorem and continuity of T', since there is $\gamma_i \in (\alpha_i, \beta_i)$ with

$$|b_i - a_i| = \mathrm{Vol}_1(T([\alpha_i, \beta_i])) = |T(\beta_i) - T(\alpha_i)| = |T'(\gamma_i)(\beta_i - \alpha_i)| = |T'(\gamma_i)||\beta_i - \alpha_i| \approx |T'(d_i)|\mathrm{Vol}_1([\alpha_i, \beta_i]),$$

where the continuity of T' is used to say that $|T'(\gamma_i)| \approx |T'(d_i)|$. The term $2(8-u) = |2(8-u)(-1)| = |T'(u)|$ in the substitution formula is therefore exactly what is needed to correct for how T distorts lengths, and is necessary for the equality

$$\int_{T([6,7])} f(x)\, dV_1(x) = \int_{[6,7]} f(T(u))|T'(u)|\, dV_1(u).$$

The key features of the previous example that will be important in our general discussion are that $[6,7]$ is an elementary domain, T is C^1 on $[6,7]$ with $T'(u) \neq 0$ on $[6,7]$ (which ensures that $f(T(u))$ will be integrable on $[6,7]$ and that our intuition about T "stretching" $[6,7]$ is valid), and that T is injective on $[6,7]$. (This ensures that we do not "double-count" parts of the integral $\int_{[1,4]} f(x)\, dV_1(x)$ when we change variables to u.)

The theorem is as follows.

Theorem 10 (Change of Variables)**.** Let $D \subset \mathbb{R}^n$ be an elementary region, and let $\Omega \subseteq \mathbb{R}^n$. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and injective on D, and that $DT(\vec{u})$ is invertible throughout D. Let $T(D)$ be the image of D under T. If $T(D) \subseteq \Omega$ and $f : \Omega \to \mathbb{R}$ is integrable on $T(D)$, then $f(T(\vec{u}))$ and $f(T(\vec{u}))|\det(DT(\vec{u}))|$ are integrable on D and

$$\int_{T(D)} f(\vec{x})\, dV_n(\vec{x}) = \int_D f(T(\vec{u}))|\det(DT(\vec{u}))|\, dV_n(\vec{u}).$$

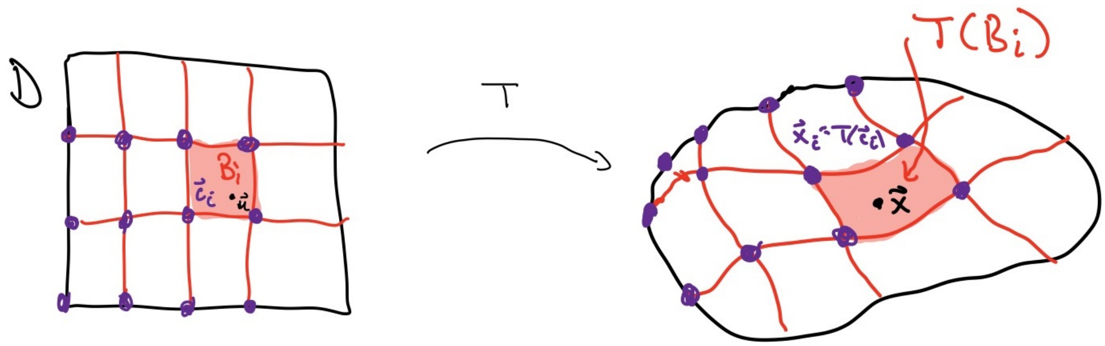
Remark 15. The above theorem will hold even if the conditions that T is injective and $DT(\vec{u})$ is invertible fail on a set $X \subseteq D$ of measure zero as long as the image $T(X)$ of this set also has measure zero. This is extremely important in applications: the images of portions of ∂D may overlap in $T(D)$, or $DT(\vec{u})$ may become zero on some small set. We'll point this out when we use it.

Remark 16. The statement of the theorem in the book (which is only in the 2- and 3- dimensional cases) is actually incorrect, as it omits the requirement that $DT(\vec{u})$ is invertible. This is actually a crucial condition for the proof, as it is used (along with the C^1 condition) to invoke the (so-called) **Inverse Function Theorem**, which allows one to infer properties of T^{-1} from those of T.

Proof. A full proof of the theorem is outside of the scope of the course (again, we need much more analysis machinery than we have). Nevertheless, we can give a sketch of the proof that should give an idea of why it is true (but perhaps won't indicate why some of the hypotheses are necessary).

To simplify the argument, assume that D is a box and f is continuous on $T(D)$. Then $f \circ T$ is continuous on D, so that $f(T(\vec{u}))|\det(DT(\vec{u}))|$ is also continuous on D because the entries of $DT(\vec{u})$ are continuous functions of $\vec{u}$, so that $|\det(DT(\vec{u}))|$ is the absolute value of a sum of products of continuous functions (and is therefore continuous). It follows that $f(T(\vec{u}))|\det(DT(\vec{u}))|$ is integrable on D. We will approximate $\int_{T(D)} f(\vec{x})\,dV_n(\vec{x})$ with Riemann sums for $\int_D f(T(\vec{u}))|\det(DT(\vec{u}))|\,dV_n(\vec{u})$.

Choose a partition $\mathcal{P}$ of D into boxes $B_1, \ldots, B_m$ and choose sample points $\vec{c}_1, \ldots, \vec{c}_m$ in a "corner" of each box.



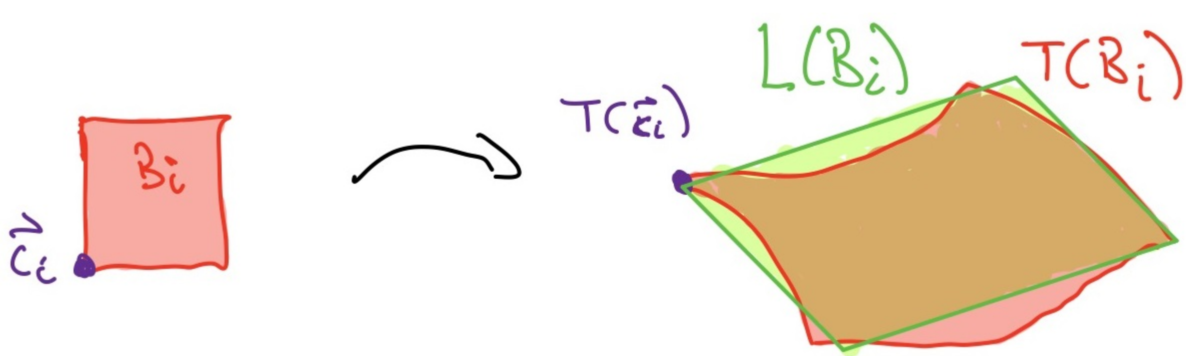
The various hypotheses on T ensure that $T(D)$ is then partitioned into the (non-box) regions $T(B_1), \ldots, T(B_m)$, which might overlap only on sets of measure zero[9] (i.e. the images of the boundaries of $B_1, \ldots, B_m$). Let $\vec{x}_i \stackrel{def}{=} T(\vec{c}_i)$ for each i.

Because T is continuous and the maximum edge length of each B_i is small (since we are eventually taking $\|\mathcal{P}\| \to 0$), for $\vec{x} = T(\vec{u}) \in T(B_i)$ we have $\|\vec{x} - \vec{x}_i\| = \|T(\vec{u}) - \vec{c}_i\|$ small, and therefore (since f is continuous) $f(\vec{x}) \approx f(\vec{x}_i)$.

Similarly, since T is differentiable at $\vec{c}_i$ we have

$$T(\vec{u}) \approx L(\vec{u}) = T(\vec{c}_i) + DT(\vec{c}_i)(\vec{u} - \vec{c}_i)$$

for $\vec{u}$ near $\vec{c}_i$, and therefore the image of the box B_i under T can be approximated by the image $L(B_i)$ of the box B_i under the affine map L.



[9]Here we are actually using a fact that could be another item in the Measure Zero Theorem: if $\vec{g} : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $Z \subset \mathbb{R}^n$ has measure zero, then $\vec{g}(Z)$ has measure zero.

Since B_i is a box with $\vec{c}_i$ a corner, there are vectors $\vec{v}_1, \ldots, \vec{v}_n$ (with $\vec{v}_i$ a scalar multiple of $\vec{e}_i$) such that

$$B_i = \{\vec{c}_i + s_1\vec{v}_1 + \cdots + s_n\vec{v}_n \;:\; s_1, \ldots, s_n \in [0,1]\}.$$

In this way we can view B_i as a translation of the parallelotope $E(\vec{v}_1, \ldots, \vec{v}_n)$, so that if A is the $n \times n$ diagonal matrix with columns $\vec{v}_1, \ldots, \vec{v}_n$, then and $\text{Vol}_n(B_i) = |\det(A)|$.

For $\vec{u} = \vec{c}_i + s_1\vec{v}_1 + \cdots + s_n\vec{v}_n \in B_i$, we have

$$L(\vec{u}) = L(\vec{c}_i) + DT(\vec{c}_i)(s_1\vec{v}_1 + \cdots + s_n\vec{v}_n) = L(\vec{c}_i) + s_1 DT(\vec{c}_i)\vec{v}_1 + \cdots + s_n DT(\vec{c}_i)\vec{v}_n.$$

Therefore $L(B_i)$ is a translation of the paralleltope $E(DT(\vec{c}_i)\vec{v}_1, \ldots, DT(\vec{c}_i)\vec{v}_n)$, so (by our expansion factor results from last quarter) we have

$$\text{Vol}_n(L(B_i)) = |\det(DT(\vec{c}_i))||\det(A)| = |\det(DT(\vec{c}_i))|\text{Vol}_n(B_i).$$

We can therefore put this all together to see that

$$\begin{aligned}
\int_{T(D)} f(\vec{x})\, dV_n(\vec{x}) &= \sum_i \int_{T(B_i)} f(\vec{x})\, dV_n(\vec{x}) \\
&\approx \sum_i \int_{T(B_i)} f(\underbrace{T(\vec{c}_i)}_{=\vec{x}_i})\, dV_n(\vec{x}) \\
&= \sum_i f(T(\vec{c}_i))\text{Vol}_n(T(B_i)) \\
&\approx \sum_i f(T(\vec{c}_i))\text{Vol}_n(L(B_i)) \\
&= \sum_i f(T(\vec{c}_i))|\det(DT(\vec{c}_i))|\text{Vol}_n(B_i) \\
&= R(f(T(\vec{u}))|\det(DT(\vec{u}))|, \mathcal{P}, \mathcal{C}).
\end{aligned}$$

As $\|\mathcal{P}\| \to 0$, the last expression approaches

$$\int_D f(T(\vec{u}))|\det(DT(\vec{u}))|\, dV_n(\vec{u}).$$

One can show (and here is where all of the difficult analysis happens) that as $\|\mathcal{P}\| \to 0$, the error in each of the "$\approx$" steps also goes to 0, so in the limit these become equalities. This gives the desired result. $\square$

Remark 17. The quantity $\det(DT(\vec{u}))$ is called the **Jacobian (determinant)** of T at $\vec{u}$. This is the convention that we will follow (and it is also the convention followed by the book), but be careful with the context, as the matrix $DT(\vec{u})$ of partial derivatives is also called the Jacobian of T at $\vec{u}$.

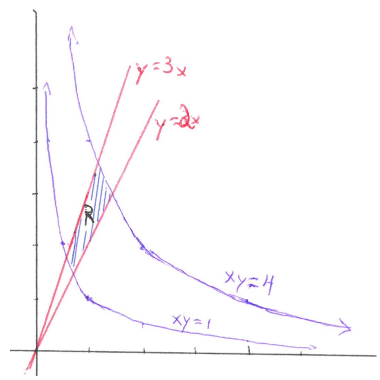
Lecture 12: More Change of Variables

Learning Objectives:

- Change variables in a multiple integral.
- Investigate the standard alternate coordinate systems in the lens of change of variables.

Example 40. Compute $\iint_R y^2\, dA(x,y)$, where $R \subset \mathbb{R}^2$ is the region in the first quadrant bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $y = 3x$ and $y = 2x$.

The region R is sketched below.



In principle it should be possible to compute this integral by splitting R into the union of (say three) elementary regions and applying Fubini's Theorem, but for a simpler approach we will first make a change of variables. We need to produce an elementary region $D \subset \mathbb{R}^2$ and a C^1, injective map $T : D \to \mathbb{R}^2$ such that $T(D) = R$ and $DT(u,v)$ is invertible for each $(u,v) \in D$. Note that $x \neq 0$ throughout R, and therefore the points $(x,y) \in R$ are characterized by the inequalities

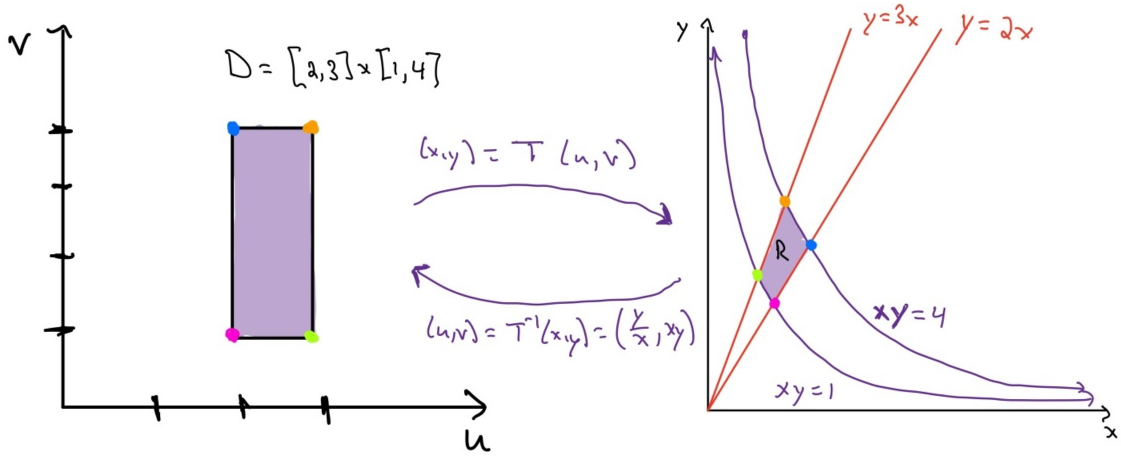
$$2x \leq y \leq 3x \quad \text{and} \quad \frac{1}{x} \leq y \leq \frac{4}{x}.$$

Rewriting these yield

$$2 \leq \frac{y}{x} \leq 3 \quad \text{and} \quad 1 \leq xy \leq 4.$$

This suggests that we should let $u = \frac{y}{x}$ and $v = xy$, and the map $T : \underbrace{[2,3] \times [1,4]}_{D} \to R$ that sends (u,v) to $T(u,v) = (x,y)$ is surjective (so that $R = T(D)$).

Above we expressed $(u,v) = (\frac{y}{x}, xy)$ as a function of (x,y), and therefore the mapping $(x,y) \mapsto (u,v) = (\frac{y}{x}, xy)$ is actually the *inverse* of T: $T^{-1}(x,y) = (u,v)$. To find T we need to express (x,y) as a function of (u,v).



To this end, note that (since $y \geq 0$ throughout R)

$$uv = \frac{y}{x}xy = y^2, \quad \text{so that } y = \sqrt{uv}.$$

We then also have $x = \frac{v}{y} = \sqrt{\frac{v}{u}}$. Because we were able to solve for (x, y) unambiguously in terms of (u, v), $T : D \to R$, $T(u, v) = (\sqrt{\frac{v}{u}}, \sqrt{uv})$ is injective as well.

Moreover, note that T is C^1 on D and

$$\det(DT(u,v)) = \det\left(\begin{bmatrix} -\frac{\sqrt{v}}{2u^{3/2}} & \frac{1}{2\sqrt{uv}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{bmatrix}\right) = -\frac{1}{4u} - \frac{1}{4u} = -\frac{1}{2u} \neq 0$$

throughout D, so that $DT(u, v)$ is invertible.

Because we can express the integrand y^2 in terms of u and v as $y^2 = (\sqrt{uv})^2 = uv$, the change of variables theorem implies that

$$\begin{aligned}
\iint\limits_R y^2 dA(x,y) &= \iint\limits_D (\sqrt{uv})^2 \left| -\frac{1}{2u} \right| dA(u,v) \\
&= \int_1^4 \int_2^3 uv \cdot \frac{1}{2u} dudv \\
&= \frac{1}{2} \int_1^4 \int_2^3 v\, dudv \\
&= \frac{1}{2} \int_1^4 v dv \\
&= \frac{15}{4}.
\end{aligned}$$

Remark 18. Note that by the chain rule, the Jacobian $\det(DT(u, v))$ satisfies, for each $(u, v) \in D$,

$$DT^{-1}(T(u,v))DT(u,v) = D(T^{-1} \circ T)(u,v) = DI(u,v) = I_2,$$

so that $DT(u, v)$ is the inverse of the matrix $DT^{-1}(x, y)$ when $(x, y) = T(u, v)$, and therefore

$$\det(DT(u,v)) = \frac{1}{\det(DT^{-1}(x,y))} \quad \text{when } (x,y) = T(u,v).$$

In the context of the previous problem, this means that we could have just computed

$$\det(DT^{-1}(x,y)) = \det\Big(\begin{bmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{bmatrix}\Big) = -2\frac{y}{x},$$

so that, using the relationship $(u,v) = (\frac{y}{x}, xy)$,

$$\det(DT(u,v)) = \frac{1}{-2\frac{y}{x}} = -\frac{1}{2u},$$

exactly as we computed above!

Remark 19. Your book uses the (fairly standard) notation

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)} \overset{notation}{=} \det(DT(\vec{u})),$$

where $\vec{x} = T(\vec{u})$. Here we are thinking of $(x_1,\ldots,x_n)$ as the component functions of T, in the sense that

$$T(u_1,\ldots,u_n) = (x_1(u_1,\ldots,u_n),\ldots,x_n(u_1,\ldots,u_n)).$$

With this notation, the content of the previous remark is that

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)} = \Big(\frac{\partial(u_1,\ldots,u_n)}{\partial(x_1,\ldots,x_n)}\Big)^{-1},$$

where the left-hand side is evaluated at $\vec{u}$ and the right-hand side is evaluated at $T(\vec{u})$.

Example 41. You will show on your homework that the Change of Variables Theorem is the missing piece that we need to give the promised generalization of the "expansion factor" interpretation of $|\det(A)|$ for $A \in M_{n\times n}(\mathbb{R})$. In particular, you will prove the following result:

> Suppose D is an elementary region in $\mathbb{R}^n$, and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible affine transformation of the form $T(\vec{x}) = A\vec{x} + \vec{b}$. Then $\mathrm{Vol}_n(T(D)) = |\det(A)|\mathrm{Vol}_n(D)$.

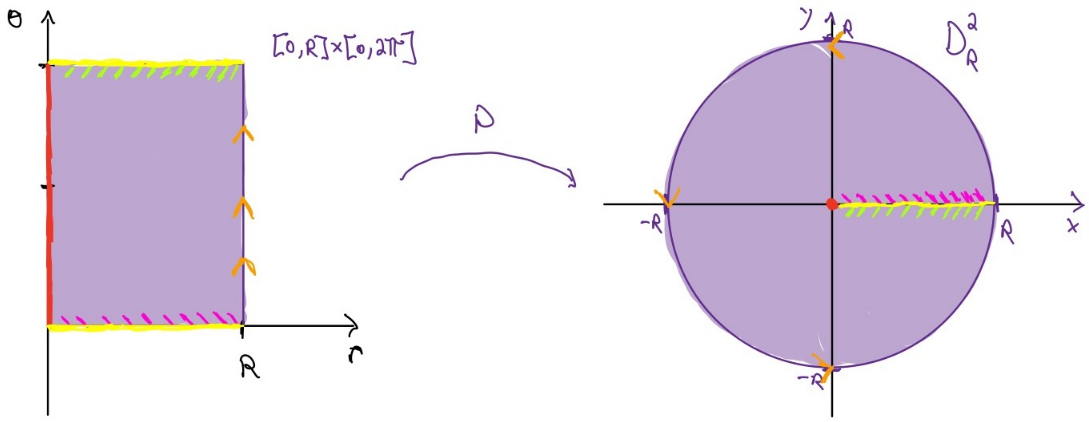
Example 42. Compute the integral $I = \iint_{D_R^2} e^{-x^2-y^2}\, dA(x,y)$, where $D_R^2 \overset{def}{=} \{(x,y) \in \mathbb{R}^2 \ : \ \|(x,y)\| \le R\}$ is the closed ball in $\mathbb{R}^2$ centered at $(0,0)$ with radius $R > 0$.

One might first attempt to approach this problem via Fubini's Theorem, but we quickly realize that, any way you slice it[10] this will be impossible because after writing the iterated integral, one would need to compute the antiderivative of e^{-x^2} (which is impossible to do using elementary functions).

However, the presence of $-(x^2+y^2)$ in the integrand and the shape of the region (determined largely by circles centered at the origin and/or lines through the origin) suggest that this integral might be more tractable if we made a change to polar coordinates.

Let $P : \mathbb{R}^2 \to \mathbb{R}^2$, $P(r,\theta) = (r\cos(\theta), r\sin(\theta))$. When $r > 0$, $P(r,\theta)$ represents $(x,y) = P(r,\theta)$ in terms of its polar coordinates (r,θ) as $(x(r,\theta), y(r,\theta)) = P(r,\theta) = (r\cos(\theta), r\sin(\theta))$.

[10] Pun intended.



Here, we note that D_R^2 is the image of the box $B = [0, R] \times [0, 2\pi]$ under P. P is not injective on B, because $P(r, 0) = (r, 0) = P(r, 2\pi)$ for each $r \in [0, R]$, and $P(0, \theta) = (0, 0)$ for every $\theta \in [0, 2\pi]$. However, this is allowable (for the purposes of the Change of Variables Theorem) because these three line segments form a set of measure zero in B, the image of these line segments forms a set of measure zero, and P is injective on the rest of B.

Note that P is C^1 on $\mathbb{R}^2$, and that

$$|\det(DP(r,\theta))| = \left|\det\left(\begin{bmatrix}\cos(\theta) & -r\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{bmatrix}\right)\right| = |r\cos^2(\theta) + r\sin^2(\theta)| = |r| = r \neq 0 \text{ for } r > 0.$$

In particular, $DP(r, \theta)$ is invertible throughout B except on the left-hand edge of B where $r = 0$. Because this line segment forms a set of measure zero, this is also allowable (for the purposes of the Change of Variables Theorem).

Because $f(x, y) = e^{-x^2-y^2}$ is continuous on $\mathbb{R}^2$ (and therefore integrable on D_R^2), the Change of Variables Theorem (and then Fubini's Theorem) implies that

$$\begin{aligned}\iint\limits_{D_R^2} f(x,y)\,dA(x,y) &= \iint\limits_{[0,R]\times[0,2\pi]} f(r\cos(\theta), r\sin(\theta))r\,dA(r,\theta)\\ &= \int_0^R \int_0^{2\pi} re^{-r^2}\,d\theta\,dr\\ &= \int_0^R 2\pi re^{-r^2}\,dr\\ &= \Big[-\pi e^{-r^2}\Big]_0^R\\ &= \pi(1 - e^{-R^2}).\end{aligned}$$

Example 43. To illustrate the importance of the injectivity condition in the Change of Variables Theorem, consider in the previous problem that it is also the case that D_R^2 is also the image of $B' = [0, R] \times [0, 3\pi]$ under P, but that P fails to be injective on $([0, R] \times [0, \pi]) \cup ([0, R] \times [2\pi, 3\pi])$, since $P(r, \theta) = P(r, \theta + 2\pi)$ for each (r, θ). If we "apply" the Change of Variables Theorem in this case we obtain the integral

$$\iint\limits_{[0,R]\times[0,3\pi]} f(r\cos(\theta), r\sin(\theta))r\,dA(r,\theta) = \int_0^R \int_0^{2\pi} re^{-r^2}\,d\theta\,dr = \frac{3\pi}{2}(1 - e^{-R^2}),$$

which is not what we had before. The issue here is

$$\iint\limits_{[0,R]\times[0,\pi]} f(r\cos(\theta), r\sin(\theta))r\,dA(r,\theta) \qquad \text{and} \qquad \iint\limits_{[0,R]\times[2\pi,3\pi]} f(r\cos(\theta), r\sin(\theta))r\,dA(r,\theta)$$

both represent

$$\iint\limits_{D_R^2,\ y\geq 0} f(x,y)\,dA(x,y),$$

so we are "double counting" this part of the integral.

Example 44. The previous example (combined with some single-variable calculus results) allows us to prove the (very famous) result that

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

To see why, define

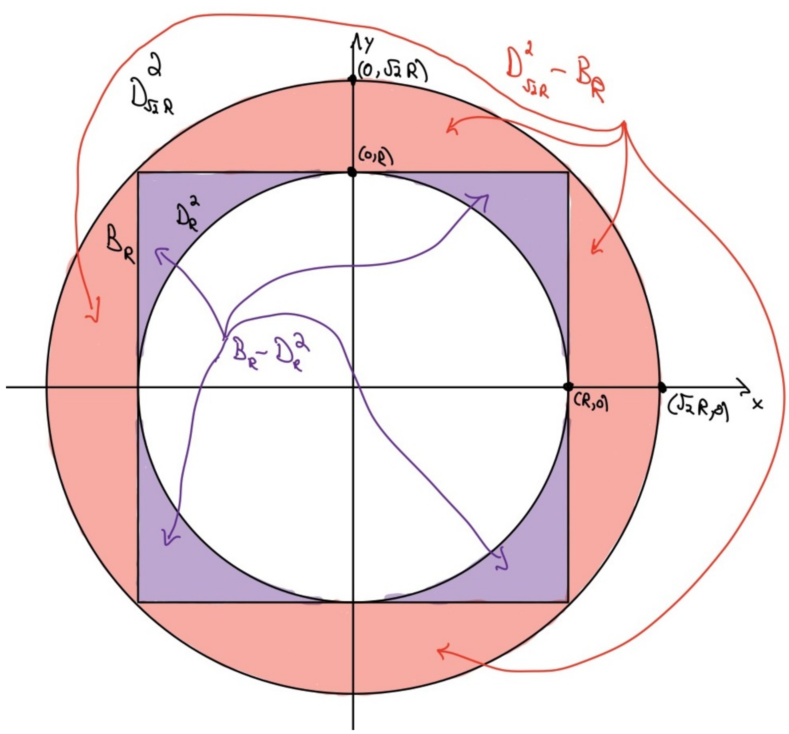
$$I_R \overset{def}{=} \int_{-R}^{R} e^{-x^2}\,dx, \quad R > 0.$$

Then $\lim\limits_{R\to\infty} I_R = I$. But Fubini's Theorem also implies that

$$(I_R)^2 = \Big(\int_{-R}^{R} e^{-x^2}\,dx\Big)\Big(\int_{-R}^{R} e^{-y^2}\,dy\Big) = \int_{-R}^{R}\int_{-R}^{R} e^{-x^2-y^2}\,dydx = \iint\limits_{B_R} e^{-x^2-y^2}\,dA(x,y),$$

where $B_R = [-R,R]\times[-R,R]$. Note that

$$D_R^2 \subset B_R \subset D_{\sqrt{2}\,R}^2.$$



Since $f(x,y) = e^{-x^2-y^2} > 0$ for each $(x,y) \in \mathbb{R}^2$, $\iint_\Omega f(x,y)\, dA(x,y) \geq 0$ for each bounded region Ω with $\partial\Omega$ of measure zero. Then we have[11]

$$\begin{aligned}
\iint\limits_{D_R^2} e^{-x^2-y^2}\, dA(x,y) &\leq \iint\limits_{D_R^2} e^{-x^2-y^2}\, dA(x,y) + \iint\limits_{B_R - D_R^2} e^{-x^2-y^2}\, dA(x,y) \\
&= \iint\limits_{B_R} e^{-x^2-y^2}\, dA(x,y) \\
&\leq \iint\limits_{B_R} e^{-x^2-y^2}\, dA(x,y) + \iint\limits_{D^2_{\sqrt{2}R} - B_R} e^{-x^2-y^2}\, dA(x,y) \\
&= \iint\limits_{D^2_{\sqrt{2}R}} e^{-x^2-y^2}\, dA(x,y).
\end{aligned}$$

Because $\iint_{D_R^2} e^{-x^2-y^2}\, dA(x,y) = \pi(1-e^{-R^2})$, this implies that

$$\pi(1-e^{-R^2}) \leq \left(\int_{-R}^{R} e^{-x^2}\, dx\right)^2 \leq \pi(1-e^{-2R^2}),$$

or rather

$$\sqrt{\pi(1-e^{-R^2})} \leq \int_{-R}^{R} e^{-x^2}\, dx \leq \sqrt{\pi(1-e^{-2R^2})}.$$

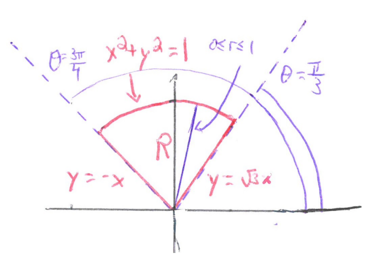
Since $\lim\limits_{R\to\infty} \sqrt{\pi(1-e^{-R^2})} = \sqrt{\pi}$ and $\lim\limits_{R\to\infty} \sqrt{\pi(1-e^{-2R^2})} = \sqrt{\pi}$, the Squeeze Theorem implies that

$$\sqrt{\pi} = \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

Note that we were able to compute this improper integral without needing to find an antiderivative for e^{-x^2} (although we did at one point find an antiderivative for xe^{-x^2}, which is much easier)!

Example 45. Compute the integral $I = \iint\limits_R y\, dA(x,y)$, where R is the region bounded below by the lines $y = \sqrt{3}x$ and $y = -x$, and above by the circle $x^2 + y^2 = 1$.

We first sketch a picture of this region:



Although we could certainly do this integral in Cartesian coordinates, we would have to split up the region. To avoid this, let's use polar coordinates. This region is a 'polar rectangle' described by the

[11] Here, $A - B \overset{def}{=} A \cap (B^c) = \{\vec{x} \in A \;:\; \vec{x} \notin B\}$.

inequalities $\frac{\pi}{3} \leq \theta \leq \frac{3\pi}{4}$ and $0 \leq r \leq 1$. We already know that the polar coordinate map satisfies the hypotheses of the Change of Variables Theorem, and the integrand $f(x, y) = y$ is continuous on $\mathbb{R}^2$ (and therefore integrable on R). We therefore have

$$\begin{aligned} I &= \iint_{[0,1]\times[\frac{\pi}{3},\frac{3\pi}{4}]} (r\sin(\theta))r\,dA(r,\theta) \\ &= \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \int_0^1 r\sin(\theta) r dr d\theta \\ &= \int_0^1 \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} r\sin(\theta) r d\theta dr \\ &= \int_0^1 r^2 \Big(-\cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{\pi}{3}\right)\Big) dr \\ &= \int_0^1 r^2 \Big(\frac{1+\sqrt{2}}{2}\Big) dr \\ &= \frac{1+\sqrt{2}}{6}. \end{aligned}$$

Lecture 13: Even More Change of Variables

Learning Objectives:

- Change variables in a multiple integral.
- Investigate the standard alternate coordinate systems in the lens of change of variables.

Today we continue our discussion of change of coordinates by doing examples in 3 dimensions. We'll focus on two special coordinate systems: cylindrical and spherical coordinates. Before we begin, let's determine the Jacobian of each of these coordinate mappings. Suppose throughout that f is integrable on a region $D^* \subset \mathbb{R}^3$.

We start with cylindrical coordinates, which corresponds to the mapping[12]

$$(r, \theta, z) \mapsto (x(r, \theta, z), y(r, \theta, z), z(r, \theta, z)),$$

with

$$x(r, \theta, z) = r\cos(\theta), \quad y(r, \theta, z) = r\sin(\theta), \quad z(r, \theta, z) = z.$$

This transformation is C^1 and the Jacobian with respect to this transformation is

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \det \begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = r \neq 0 \text{ except when } r = 0,$$

and therefore if D is an elementary region in $\mathbb{R}^3$ (with $r \geq 0$) such that $(r, \theta, z) \mapsto (x, y, z)$ is an injective (except possibly on a set of measure zero) map of D onto D^*, we have

$$\iiint\limits_{D^*} f(x, y, z)\, dV(x, y, z) = \iiint\limits_{D} f(r\cos(\theta), r\sin(\theta), z) r \, dV(r, \theta, z).$$

The fact that the Jacobian of this transformation was r (which is the same as for polar coordinates) is not surprising; after all, cylindrical coordinates in $\mathbb{R}^3$ is what we get when we use polar coordinates to replace x and y (leaving z alone).

For spherical coordinates, we consider the mapping

$$(\rho, \phi, \theta) \mapsto (x(\rho, \phi, \theta), y(\rho, \phi, \theta), z(\rho, \phi, \theta)),$$

with

$$x(\rho, \phi, \theta) = \rho\cos(\theta)\sin(\phi), \quad y(\rho, \phi, \theta) = \rho\sin(\theta)\sin(\phi), \quad z(\rho, \phi, \theta) = \rho\cos(\phi).$$

[12]Here, to be consistent with the notation used for both rectangular coordinates in $\mathbb{R}^3$ and cylindrical coordinates in $\mathbb{R}^3$, we will use z to denote the third coordinate in both settings. This should not cause any confusion as long as we are careful to consider the context!

The Jacobian for this transformation is

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \det \begin{bmatrix} \cos(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) & -\rho\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) & \rho\cos(\theta)\sin(\phi) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{bmatrix} \\
&= \rho^2\sin(\phi)\det \begin{bmatrix} \cos(\theta)\sin(\phi) & \cos(\theta)\cos(\phi) & -\sin(\theta) \\ \sin(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) & \cos(\theta) \\ \cos(\phi) & -\sin(\phi) & 0 \end{bmatrix} \\
&= \rho^2\sin(\phi) \neq 0 \text{ except when } \rho = 0 \text{ or } \phi = k\pi,\ k \in \mathbb{Z},
\end{aligned}$$

and therefore if D is an elementary region in $\mathbb{R}^3$ (with $\rho \geq 0$ and $0 \leq \phi \leq \pi$) such that $(\rho,\phi,\theta) \mapsto (x,y,z)$ is an injective (except possibly on a set of measure zero) map of D onto D^*, we have

$$\iiint_{D^*} f(x,y,z)\,dV(x,y,z) = \iiint_{D} f(\rho\sin(\theta)\sin(\phi),\rho\cos(\theta)\sin(\phi),\rho\cos(\phi))\rho^2\sin(\phi)\,dV(\rho,\phi,\theta).$$

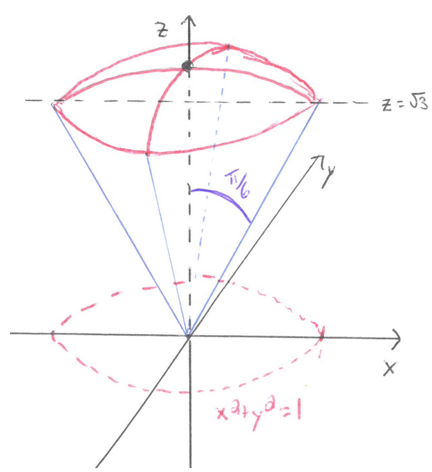
Here (and henceforth), when using spherical coordinates we will require that $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

When changing to polar (or cylindrical or spherical) coordinates, it is often the case that it is easier to go directly to an iterated integral (via Fubini's Theorem), as the highly geometric nature of the new variables make considering the exact shape of the region in the $r\theta$-plane (or $r\theta z$-space or $\rho\phi\theta$-space) somewhat superfluous.

Example 46. Set up the integral of $f(x,y,z) = x + y$ over the SnoCone-shaped region E, which is bounded below by the cone $z = \sqrt{3}\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2 = 4$, in

- Cylindrical coordinates in the order $dzdrd\theta$
- Cylindrical coordinates in the order $drdzd\theta$
- Spherical coordinates in the order $d\rho d\phi d\theta$

We want to set up $\iiint_E f(x,y,z)dV$, where E is sketched below:



Since r and θ are the middle and outer variables, in this case the outer two integrals just describe (in polar coordinates) the 'shadow' of E in the xy-plane. This shadow is the region which is between the origin and the shadow of the intersection of the sphere and the cone. This intersection is found by eliminating z from the two equations, which gives

$$3(x^2+y^2) = 4 - x^2 - y^2, \quad \text{or rather} \quad x^2+y^2 = 1,$$

the circle of radius 1 (in the xy-plane) centered at the origin. Therefore, we have $0 \le \theta \le 2\pi$ and $0 \le r \le 1$.

For the bounds for z, we simply note that, for each fixed r and θ, z runs from the cone $z = \sqrt{3}\sqrt{x^2+y^2} = \sqrt{3}r$ to the top half of the sphere: $z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$. Hence, our integral is (after applying Fubini's Theorem)

$$\iiint_E (x+y)\, dV(x,y,z) = \int_0^{2\pi}\int_0^1\int_{\sqrt{3}r}^{\sqrt{4-r^2}} (r\cos(\theta)+r\sin(\theta))r\, dzdrd\theta.$$

Let's now set up this integral in cylindrical coordinates with respect to the order $drdzd\theta$. Here have $0 \le \theta \le 2\pi$ and that $0 \le z \le 4$. For fixed θ and z, r will run from 0 (corresponding to the z axis) out until we hit the boundary of E. Here we see that the upper bound changes depending on the value of z. Indeed, when $0 \le z \le \sqrt{3}$ we have $0 \le r \le \frac{z}{\sqrt{3}}$, while if $\sqrt{3} \le z \le 2$ we have $0 \le r \le \sqrt{4-z^2}$. Therefore, the integral can also be written as

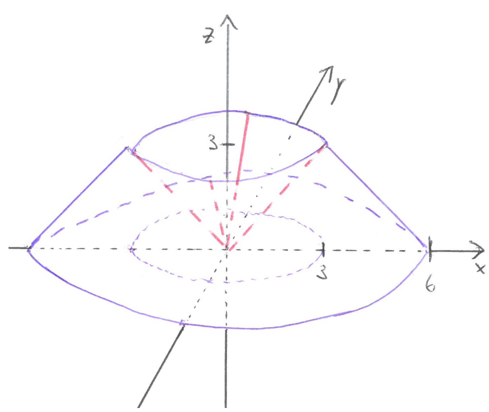
$$\begin{aligned}\iiint_E (x+y)\, dV(x,y,z) &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{\frac{z}{\sqrt{3}}} (r\cos(\theta)+r\sin(\theta))r\, drdzd\theta \\ &\quad + \int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_0^{\sqrt{4-z^2}} (r\cos(\theta)+r\sin(\theta))r\, drdzd\theta.\end{aligned}$$

Now we set up this integral in spherical coordinates as well, using the order $d\rho d\phi d\theta$. As above, we still have $0 \le \theta \le 2\pi$. For ϕ, we note that, for θ fixed, ϕ runs from 0 until it hits the cone at $\phi = \frac{\pi}{6}$. Once we fix θ and ϕ, we see that ρ runs from 0 (i.e. the origin) until it hits the sphere, so that $0 \le \rho \le 2$. Therefore, in spherical coordinates this integral becomes

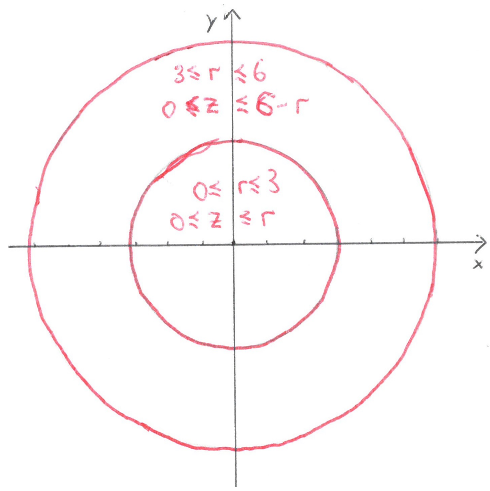
$$\iiint_E (x+y)\, dV(x,y,z) = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2 (\rho\sin(\phi)\cos(\theta)+\rho\sin(\phi)\sin(\theta))\rho^2\sin(\phi)\, d\rho d\phi d\theta.$$

Example 47. Let E be the region in $\mathbb{R}^3$ which is bounded below by the xy-plane, and above by the cones $z = \sqrt{x^2+y^2}$ and $z = 6 - \sqrt{x^2+y^2}$. Set up the integral of a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ over E in (a) cylindrical coordinates in the order $dzdrd\theta$, (b) spherical coordinates in the order $d\rho d\theta d\phi$

We sketch this region below:



(a) Here, note that we have $0 \le \theta \le 2\pi$ and $0 \le r \le 6$. The bounds for z are a bit trickier; when $0 \le r \le 3$ (and $0 \le \theta \le 2\pi$), z runs from 0 (i.e. the xy-plane) until it hits the cone $z = \sqrt{x^2+y^2} = r$, so that $0 \le z \le r$. On the other hand, if $3 \le r \le 6$ (and $0 \le \theta \le 2\pi$), then z runs from 0 (i.e. the xy-plane) to the cone $z = 6 - \sqrt{x^2+y^2} = 6 - r$, so that $0 \le z \le 6 - r$.



Hence, we have

$$\iiint\limits_E f(x,y,z)dV = \int_0^{2\pi}\int_0^3\int_0^r f(x(r,\theta,z),y(r,\theta,z),z(r,\theta,z))r\,dzdrd\theta + \int_0^{2\pi}\int_3^6\int_0^{6-r} f(x(r,\theta,z),y(r,\theta,z),z(r,\theta,z))r\,dzdrd\theta.$$

(b) Since the region is below the cone $z=\sqrt{x^2+y^2}$ but above the xy-plane, one sees that $\frac{\pi}{4}\le\phi\le\frac{\pi}{2}$. Also, we still have $0\le\theta\le 2\pi$. For every fixed ϕ, note that ρ runs from 0 (i.e. the origin) until it hits the cone $z=6-\sqrt{x^2+y^2}$. Substituting in our spherical formulas for x, y, and z yields

$$\rho\cos(\phi)=6-\rho\sin(\phi),\quad \text{or rather}\quad \rho=\frac{6}{\cos(\phi)+\sin(\phi)}.$$

Therefore, we have

$$\iiint\limits_E f(x,y,z)dV = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\frac{6}{\cos(\phi)+\sin(\phi)}} f(x(\rho,\phi,\theta),y(\rho,\phi,\theta),z(\rho,\phi,\theta))\rho^2\sin(\phi)\,d\rho d\theta d\phi.$$

Example 48. Without resorting to explicit calculation, show that

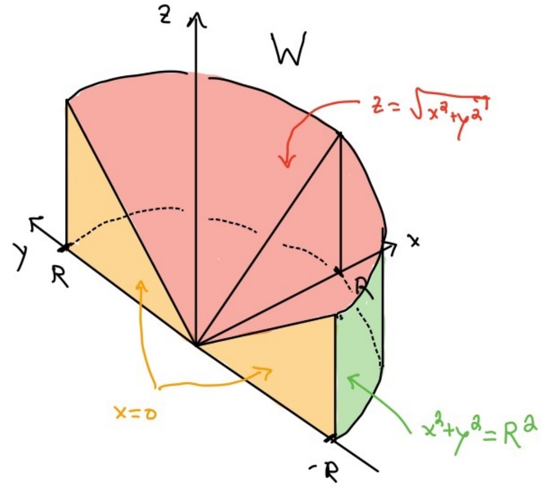
$$\int_{-\pi/2}^{\pi/2}\int_0^1\int_z^R \theta r\,drdzd\theta = \int_{-\pi/2}^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{R/\sin(\phi)} \theta\,\rho^2\sin(\phi)\,d\rho d\phi d\theta$$

Note that each of these two iterated integrals represent the triple integral of

$$f(x,y,z)=\theta=\begin{cases}\arctan(\frac{y}{x}) & \text{if } x>0,\\ \frac{\pi}{2} & \text{if } x=0,\ y>0,\\ -\frac{\pi}{2} & \text{if } x=0,\ y<0\end{cases}$$

over regions in $\mathbb{R}^3$ with $x\ge 0$ (since $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$). The first integral appears to be in terms of cylindrical coordinates, and the second appears to be in terms of spherical coordinates. We argue that the region of integration is secretly the same for both integrals.

For the first integral, we note that the condition that $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ implies that the region lies on the side of the yz-plane where $x\ge 0$. The condition that $0\le z\le 1$ implies that this region also lies between the xy-plane (i.e. $z = 0$) and the plane $z = 1$. Finally, the condition that $z \le r \le R$ can be expressed as $z \le \sqrt{x^2 + y^2} \le R$. The inequality $z \le \sqrt{x^2 + y^2}$ indicates that the region lies below the cone $z = \sqrt{x^2 + y^2}$, and the inequality $\sqrt{x^2 + y^2} \le R$ indicates that the region is enclosed by the circular cylinder $x^2 + y^2 = R^2$. We sketch this region (call it W) below:



We can attempt to repeat this process for the second integral, but the bound $\rho \le R/\sin(\phi)$ appears to be a little mysterious. Rather than reverse-engineer this bound, let's try to represent W in spherical coordinates directly and then compare.

First note that, since θ has the same meaning in both cylindrical and spherical coordinates, we still have $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. For each fixed θ, it appears that ϕ will run from $\phi = \frac{\pi}{4}$ (describing points that lie on the cone $z = \sqrt{x^2 + y^2}$) to $\phi = \frac{\pi}{2}$ (describing points in the xy-plane), so that $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$. For fixed θ and ϕ, ρ will run from 0 (i.e. the origin) until we hit the cylinder $x^2 + y^2 = R^2$. We therefore see that the upper bound for ρ will satisfy

$$R^2 = x^2 + y^2 = x^2 + y^2 + z^2 - z^2 = \rho^2 - \rho^2 \cos^2(\phi) = \rho^2 \sin^2(\phi), \quad \text{so that } \rho = \frac{R}{\sin(\phi)}.$$

Therefore we have $0 \le \rho \le \frac{R}{\sin(\phi)}$, and the resulting triple integral in spherical coordinates is exactly the one given in the problem statement. Because these two iterated integrals represent the triple integral of the same function over the same region in $\mathbb{R}^3$, they have the same value.

Lecture 14: Curves

Learning Objectives:

- Parametrize a curve with a path.
- Interpret the derivative of a path as the vector tangent to a curve.
- Compute the length of a curve.

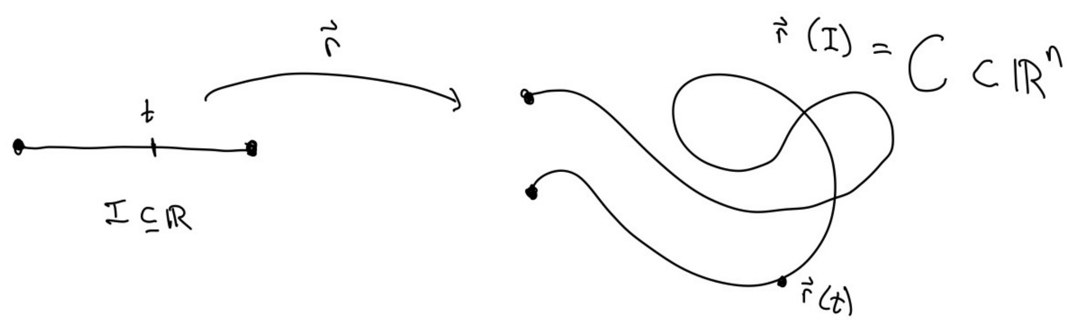
Now that we know how to integrate functions over regions in $\mathbb{R}^n$, we shift gears to ask about integrating over other types of subsets of $\mathbb{R}^n$. For instance, can we integrate over a *curve* in $\mathbb{R}^2$ or $\mathbb{R}^3$? What about a *surface* in $\mathbb{R}^3$? What about higher-dimensional analogs in $\mathbb{R}^n$? The answer to these questions turns out to be affirmative, and the key to understanding how to make sense out of such integrals lies in parameterization. That is, we can integrate over a curve by parameterizing the curve and then using a 'change-of-variables'-type formula to get an integral over an interval in the real line. Similarly, we will integrate over surfaces by parameterizing them with regions in $\mathbb{R}^2$ and then using a formula that looks suspiciously like 'change-of-variables'.

We begin this process by giving some definitions.

Definition 12. A **path** is a continuous function $\vec{r} : I \to \mathbb{R}^n$ for some interval $I \subseteq \mathbb{R}$. We call $C \subset \mathbb{R}^n$ a **(parametric) curve** if C is a image of a path $\vec{r} : I \to \mathbb{R}^n$ that is injective except possibly on a finite set. Such a path $\vec{r}$ is said to **parametrize** C, and in this case we say that C has **parametric equations**

$$\vec{r}(t) = (x(t), y(t)) \quad (\text{if } n = 2) \qquad \text{or} \qquad \vec{r}(t) = (x(t), y(t), z(t)) \quad (\text{if } n = 3),$$

and similarly in higher dimensions. If $\vec{r}$ is C^1, then we call C a C^1 **curve**. If $\vec{r}$ can be taken C^1 such that $\vec{r}\,'(t) \neq \vec{0}$ for every $t \in I$, then we say that C is **smooth**. If C is the union of a finite number of smooth curves, then we call C **piecewise smooth**.



Last quarter, we thought of the path $\vec{r}(t)$ which parameterized a curve C as describing the motion of a particle on C whose position at time t is $\vec{r}(t)$. This will be a useful viewpoint going forward.

If $\vec{r}: I \to \mathbb{R}^n$ is differentiable at t, then the derivative is an $n \times 1$ matrix (i.e. a column vector) just like $\vec{r}$, and we therefore use the notation

$$\vec{r}\,'(t) \overset{not.}{=} D\vec{r}(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

Thinking of $\vec{r}(t)$ as the position function of a particle, it is sometimes convenient to call $\vec{v}(t) \overset{def}{=} \vec{r}\,'(t)$ the **velocity** of the particle (since it describes the direction and speed of travel at time t). Indeed, $\|\vec{r}\,'(t)\| = \|\vec{v}(t)\|$ is the **speed** at which the particle travels at time t. Hence, unless $\vec{v}(t) = 0$ (i.e. the particle is at rest), we have

$$\vec{v}(t) = \|\vec{v}(t)\| \Big(\frac{1}{\|\vec{v}(t)\|} \vec{v}(t) \Big) = (speed)(direction).$$

In the same vein (and if $\vec{r}\,'$ is differentiable at t) then $\vec{a}(t) \overset{def}{=} \vec{v}\,'(t) = \vec{r}\,''(t)$ is the **acceleration** of the particle at time t.

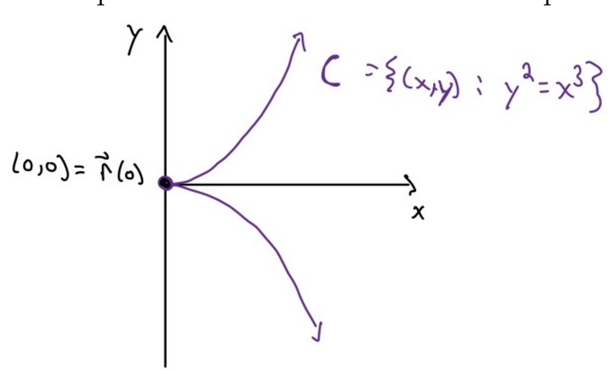
Remark 20. Note that $\vec{v}(t) = D\vec{r}(t) = D\vec{r}(t)[1]$, so that the velocity of a particle whose position is given by $\vec{r}(t)$ is given by the image under the linear transformation $D\vec{r}(t) : \mathbb{R}^1 \to \mathbb{R}^n$ of the vector $[1]$. This ties velocity into our understanding of the derivative as a linear transformation which acts on **tangent vectors**, in the sense that if $\vec{r}$ parametrizes a curve C, then $\vec{r}\,'(t_0) = D\vec{r}(t_0)[1]$ is tangent to the curve C at $\vec{r}(t_0)$.

Example 49. For $a \in \mathbb{R}$, compute parametric equations for the line tangent to the curve $\vec{r}(t) = (at, \cos(t), \sin(t))$ at the point $\vec{r}(\frac{3\pi}{2})$.

The tangent line to the curve contains the point $\vec{r}(\frac{3\pi}{2}) = \begin{bmatrix} \frac{3\pi}{2}a \\ 0 \\ -1 \end{bmatrix}$ and is parallel to the vector $\vec{r}\,'(\frac{3\pi}{2}) = \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}$. Hence, parametric equations for the line are

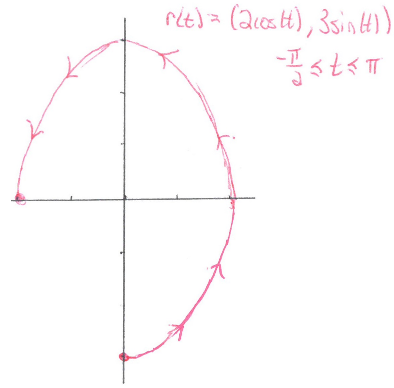
$$\vec{L}(s) = \vec{r}\Big(\frac{3\pi}{2}\Big) + s\vec{r}\,'\Big(\frac{3\pi}{2}\Big) = \Big(\Big(\frac{3\pi}{2} + s\Big)a, s, -1\Big).$$

Example 50. The term "smooth" does indeed go beyond mere "C^1", since we can have C^1 curves that have corners or curps. For an example, consider the curve C given by the equation $y^2 = x^3$. The curve C has a cusp at $(0,0)$, but C is still a C^1 curve because we can parametrize C with the C^1 path $\vec{r} : \mathbb{R} \to \mathbb{R}^2$, $\vec{r}(t) = (t^2, t^3)$. Note that when $t = 0$ (which corresponds to the point $\vec{r}(0) = (0,0)$ where C has a cusp), we have $\vec{r}\,'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This illustrates how features like cusps can be accounted for by C^1 paths when the derivatives of these paths are allowed to vanish at a point.



Example 51. Parameterize (with the counterclockwise **orientation**, or direction) the portion of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ which lies in quadrants 4, 1, and 2. Also sketch the indicated path.

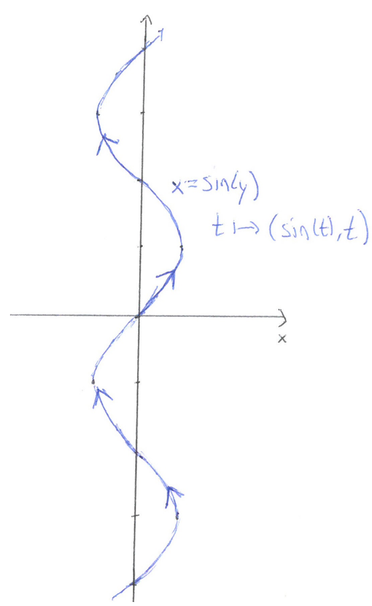
The path $\vec{r}(t) = (2\cos(t), 3\sin(t))$, $-\frac{\pi}{2} \leq t \leq \pi$ parameterizes the portion of ellipse in the direction indicated. The **endpoints** of the path are $\vec{r}(-\frac{\pi}{2}) = (0, -3)$ and $\vec{r}(\pi) = (-2, 0)$.



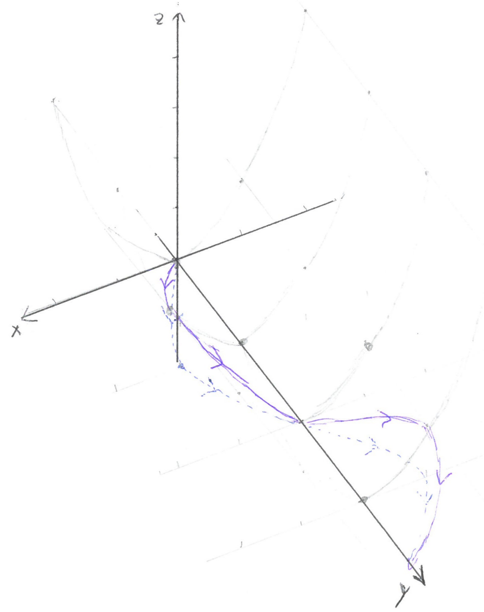
Note that $\vec{x}(s) = (2\cos(2s), 3\sin(2s))$, $-\frac{\pi}{4} \leq s \leq \frac{\pi}{2}$ also parameterizes the same portion of ellipse in the same direction. If we think about this as describing the motion of a particle, it seems that this particle is moving twice as fast as the previous one. We will make this precise in a minute.

Remark 21. The notion of orientation will be made more precise later on in the course when we discuss line integrals.

Example 52. The path $\vec{r}(t) = (\sin(t), t, \sin^2(t))$, $-\infty < t < +\infty$ traces out a curve in $\mathbb{R}^3$. To determine what this curve looks like, note that the projection of this curve in the xy-plane is all points of the form $(\sin(t), t)$, which is exactly the graph of the equation $x = \sin(y)$:



We also notice that since $z(t) = \sin^2(t) = (x(t))^2$, this curve lives on the parabolic cylinder $z = x^2$. A sketch of this curve is

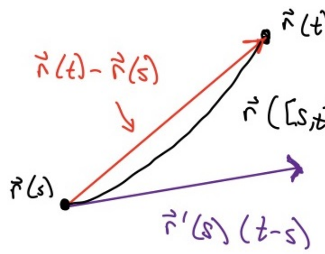


Let's think of this curve as describing the motion of a particle. Since the y-coordinate of the position increases as t increases, we see that the **orientation** of this curve (i.e. what direction we are traveling in) is as pictured above (with arrows).

Arclength

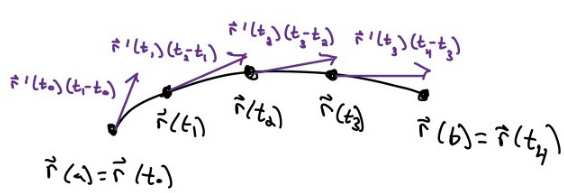
The notion of 'speed' allows us to measure the length of a C^1 path $\vec{r} : [a, b] \to \mathbb{R}^n$. Note that (by differentiability) if $s, t \in [a, b]$ with $s < t$ are close to each other, then from differentiability of $\vec{r}$ we would expect that

$$\vec{r}(t) - \vec{r}(s) \approx D\vec{r}(s)[t - s], \quad \text{so that} \quad \|\vec{r}(t) - \vec{r}(s)\| \approx \|\vec{r}\,'(s)\|(t - s).$$



On the other hand, we expect that the length of the portion of C between $\vec{r}(s)$ and $\vec{r}(t)$ should be approximately $\|\vec{r}(t) - \vec{r}(s)\|$. Therefore, if $a = t_0 < t_1 < \ldots < t_m = b$ gives a partition $\mathcal{P} = \{[t_0, t_1], [t_1, t_2], \ldots, [t_{m-1}, t_m]\}$ of $[a, b]$, then we would expect that the length of $\vec{r}$ should be

$$\sum_{i=1}^{m} \text{Length of } \vec{r}([t_{i-1}, t_i]) \approx \sum_{i=1}^{m} \|\vec{r}\,'(t_{i-1})\|(t_i - t_{i-1}) = \sum_{i=1}^{m} \|\vec{r}\,'(t_{i-1})\| \text{Vol}_1([t_{i-1}, t_i]) \overset{\|\mathcal{P}\| \to 0}{\longrightarrow} \int_a^b \|\vec{r}\,'(t)\| \, dt.$$



Therefore it is reasonable to define the **length** of $\vec{r}$ to be

$$\text{Length of } \vec{r} \stackrel{def}{=} \int_a^b \|\vec{r}\,'(t)\|\, dt.$$

If $\vec{r}$ parametrizes a curve C, then we define the **length** of C to be the length of $\vec{r}$. That is,

$$\text{Length of } C \stackrel{def}{=} \int_a^b \|\vec{r}\,'(t)\|\, dt.$$

Remark 22. The above definition of the length of a curve should make you uneasy because, in general, there are infinitely many different ways to parametrize a curve C. For this definition to be meaningful, we need to make sure that the "length" we compute does not depend on the parametrization that we use. To see this, suppose that C is a smooth C^1 curve with two different parametrizations

$$\vec{r} : [a,b] \to \mathbb{R}^n \quad \text{and} \quad \vec{s} : [c,d] \to \mathbb{R}^n,$$

each with non-vanishing derivative, related to each other[13] by $\vec{s} = \vec{r} \circ \tau$, where $\tau : [c,d] \to [a,b]$ is a C^1, bijective function with $\tau(u) \neq 0$ for each $u \in [c,d]$. Then we apply the Change of Variables Theorem and the Chain Rule to obtain

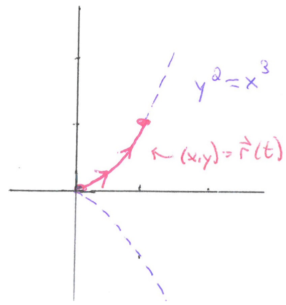
$$\begin{aligned}
\int_a^b \|\vec{r}\,'(t)\|\, dt &= \int_{\tau([c,d])} \|\vec{r}\,'(t)\|\, dV_1(t) \\
&= \int_{[c,d]} \|\vec{r}\,'(\tau(u))\| |\tau'(u)|\, dV_1(u) \\
&= \int_c^d \|\tau'(u)\vec{r}\,'(\tau(u))\|\, du \\
&= \int_c^d \|D\vec{r}(\tau(u))D\tau(u)\|\, du \\
&= \int_c^d \|D(\vec{r} \circ \tau)(u)\|\, du \\
&= \int_c^d \|D\vec{s}(u)\|\, du \\
&= \int_c^d \|\vec{s}\,'(u)\|\, du,
\end{aligned}$$

and therefore we obtain the same value for the length of C regardless of which C^1 path with non-vanishing derivative we use to parametrize C.

Example 53. Let's compute the length of the arc C of the graph of $y^2 = x^3$ in $\mathbb{R}^2$ between $(x,y) = (0,0)$ and $(x,y) = (1,1)$.

This curve is graphed in red below:

[13] Indeed, we actually must have $\tau = \vec{r}^{-1} \circ \vec{s}$. This map is bijective is C^1, bijective, and has non-zero derivative throughout $[c,d]$. The proof of this requires a bit of analysis.



Our first step is to parameterize this portion of curve. There are many correct ways to do this, but we will use

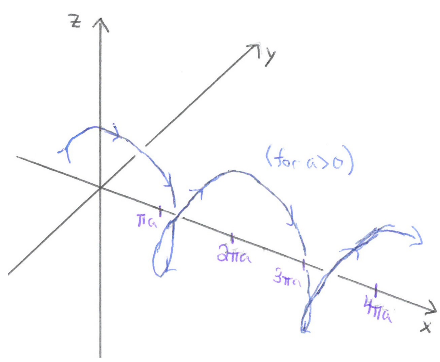
$$\vec{r}(t) \stackrel{def}{=} (t, t^{\frac{3}{2}}), \quad 0 \leq t \leq 1.$$

We therefore can write

$$\text{Length of } C = \int_0^1 \|\vec{r}\,'(t)\| dt = \int_0^1 \sqrt{1^2 + \left(\frac{3}{2}\sqrt{t}\right)^2}\, dt = \int_0^1 \sqrt{1 + \frac{9}{4}t}\, dt = \frac{8}{27}\left(1 + \frac{9}{4}t\right)^{\frac{3}{2}}\Big|_0^1 = \frac{13^{\frac{3}{2}} - 8}{27.}$$

Example 54. For $a \geq 0$, consider the portion of helix parameterized by $\vec{r}(t) = (at, \cos(t), \sin(t))$, $0 \leq t \leq 2\pi$. Sketch the curve, and prove that its length $L(a)$ satisfies $\lim_{a \to +\infty} (L(a) - 2\pi a) = 0$. Given that $\|\vec{r}(0) - \vec{r}(2\pi)\| = 2\pi a$, discuss the geometric significance of this computation.

This curve is one revolution of a helix around the x-axis with 'spacing' a between coils:



We can therefore compute that the length of this curve is

$$\begin{aligned} L(a) &= \int_0^{2\pi} \|\vec{r}\,'(t)\| dt \\ &= \int_0^{2\pi} \sqrt{(a)^2 + (-\sin(t))^2 + (\cos(t))^2 dt} \\ &= \int_0^{2\pi} \sqrt{a^2 + 1} dt \\ &= 2\pi\sqrt{a^2+1}. \end{aligned}$$

Therefore

$$\lim_{a \to +\infty} (L(a) - 2\pi a) = 2\pi \lim_{a \to +\infty} \sqrt{a^2+1} - a = 2\pi \lim_{a \to +\infty} \frac{1}{\sqrt{a^2+1} + a} = 0.$$

Since $2\pi a$ is the (straight-line) distance from the starting-point of the coil to the ending-point of the coil, this computation shows that if you stretch the coil out far enough, the arc length becomes as close to the straight-line distance as we'd like (even though it wraps once around the cylinder $y^2 + z^2 = 1$!).

Remark 23. If C is a smooth C^1 curve in $\mathbb{R}^n$ and $f : C \to \mathbb{R}$ is continuous, then the above Riemann sum argument for arclength can be adjusted motivate the definition of the **scalar line integral** of f over C as

$$\int_C f\, ds \stackrel{def}{=} \int_a^b f(\vec{r}(t))\|\vec{r}\,'(t)\|\, dt,$$

where $\vec{r} : [a, b] \to \mathbb{R}^n$ is a C^1 path (with non-vanishing derivative) that parametrizes C. On the homework you will explore this definition (and show that it does not depend on the path we use to parametrize C). Here ds represents an "infinitesimal change in length" of the curve C.

Example 55. Wally the worm lives at $(1, 0, 0)$ inside of a giant ball of cheese which has density $d(x, y, z) = x^2 + y^2 + z^2$ at (x, y, z). One day, Wally eats his way through the cheese along the helix C parameterized by $\vec{r}(t) = (\cos(t), \sin(t), t)$, $0 \le t \le 4\pi$. Much much cheese-mass does Wally eat?

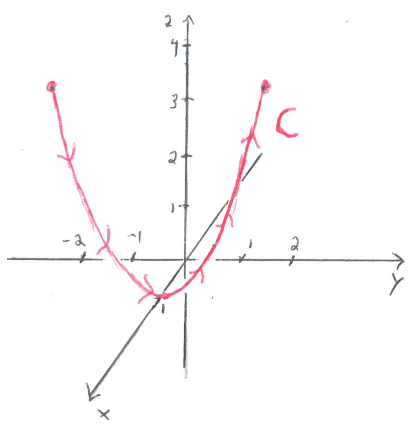
Wally eats

$$\begin{aligned}\int_C (x^2 + y^2 + z^2) ds &= \int_0^{4\pi} (\cos^2(t) + \sin^2(t) + t^2) \underbrace{\sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}}_{=\|\vec{r}\,'(t)\|} dt \\ &= \sqrt{2} \int_0^{4\pi} 1 + t^2 dt \\ &= \sqrt{2}\Big(4\pi + \frac{64\pi^3}{3}\Big)\end{aligned}$$

units of cheese-mass. Doesn't that sound delicious?

Example 56. Compute $\displaystyle\int_C y e^{z^2} ds$, where C is the curve parameterized by $\vec{r}(t) = (1, t, t^2)$, $-2 \le t \le 2$.

The curve C is the portion of the parabola $z = y^2$, $-2 \le y \le 2$ in the plane $x = 1$. We sketch this below:



Since this curve is symmetric across the xz-plane (i.e. if $(x, y, z) \in C$, then $(x, -y, z) \in C$), and since $y e^{z^2}$ is odd in y, we immediately see that

$$\int_C y e^{z^2}\, ds = 0$$

by symmetry. Of course, in terms of our parametrization this gives

$$\int_C y e^{z^2}\, ds = \int_{-2}^{2} t e^{t^4} \|\vec{r}\,'(t)\|\, dt = \int_{-2}^{2} t e^{t^4} \sqrt{1 + 4t^2}\, dt,$$

which is the integral of an odd function over an interval of the form $[-a, a]$ for $a \ge 0$ (which is 0).

Lecture 15: Surfaces

Learning Objectives:

- Parametrize a surface.
- Compute the normal vector to the surface.
- Extract information about a surface by analyzing the normal vector arising from a parametrization.
- Compute the surface area of a smooth surface.

The parametrization techniques and results that we discussed for (parametric) curves in $\mathbb{R}^n$ can be generalized to handle (parametric) surfaces. Our interest in surfaces will be limited to surfaces in $\mathbb{R}^3$, but know that one could continue to expand these ideas to more general settings. We will give an idea of these more general settings later, once we have more intuition for what we expect to hold. Some of the quirks in the definition we give are intended to avoid certain "degenerate" situations.

Definition 13. We call $S \subset \mathbb{R}^3$ a (parametric) **surface** if S is the image of a continuous function $\vec{X} : D \to \mathbb{R}^3$ that is injective except possibly on ∂D, where $D \subseteq \mathbb{R}^2$ is an elementary region (perhaps with some of its boundary points removed). Such a function $\vec{X}$ is said to **parametrize** S, and in this case we say that S has **parametric equations**

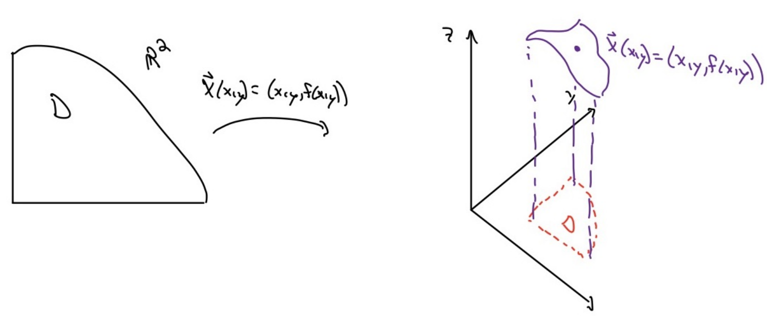
$$\vec{X}(s,t) = (x(s,t), y(s,t), z(s,t)).$$

If $\vec{X}$ is C^1, then we call S a C^1 **surface**.

Remark 24. We will also have a notion of *smooth* surface, but the definition requires slightly more machinery than what was needed to define smooth curves. We will revisit this shortly.

Example 57. Suppose that $D \subseteq \mathbb{R}^2$ is an elementary region and that $f : D \to \mathbb{R}$ is a C^1 function. Then the graph of f is a C^1 surface with parametrization

$$\vec{X} : D \to \mathbb{R}^3, \quad \vec{X}(x,y) = (x, y, f(x,y)).$$



One can permute the variables as well. For example,

$$\vec{X} : \{(x, z) \ : \ x^2 + z^2 \leq R^2, \ 0 \leq x \leq R\} \to \mathbb{R}^3, \qquad \vec{X}(x, z) = (x, -\sqrt{R^2 - x^2 - z^2}, z)$$

parametrizes the portion S of the sphere of radius R centered at $(0, 0, 0)$ that lies in the region where $y \leq 0$ and $x \geq 0$, thinking of this surface as the graph of y as a function of x and z.

Example 58. Alternate coordinate systems can also be useful when parametrizing surfaces. For example, the portion of the cone $z = \sqrt{x^2 + 4y^2}$ with $y \geq 0$ can be expressed in cylindrical coordinates as the collection of (r, θ, z) with $0 \leq r$, $0 \leq \theta \leq \pi$, and $z = \sqrt{r^2 \cos^2(\theta) + 4r^2 \sin^2(\theta)} = r\sqrt{1 + 3\sin^2(\theta)}$, which implies that we can parametrize it as

$$\vec{Y} : \{(r, \theta) \ : \ 0 \leq r, \ 0 \leq \theta \leq \pi\} \to \mathbb{R}^3, \qquad \vec{Y}(r, \theta) = (r\cos(\theta), r\sin(\theta), r\sqrt{1 + 3\sin^2(\theta)}).$$

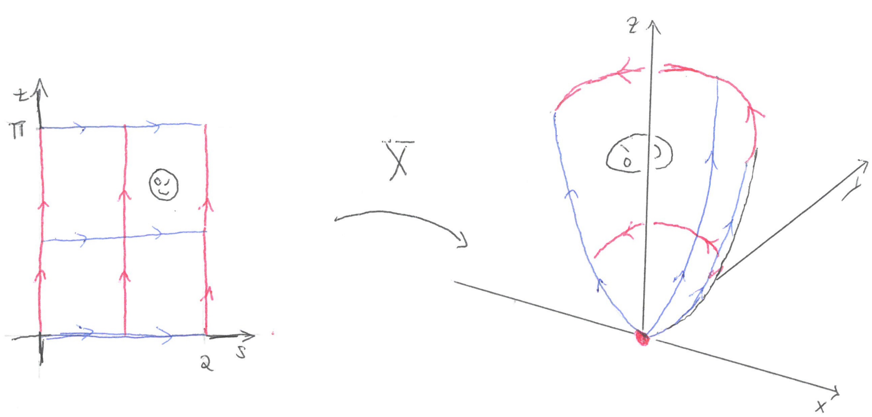
Note that the parametrization $\vec{Y}$ is actually C^1, so that the cone is a C^1 surface. (Don't worry: the cone will fail to be *smooth* at the point at which you think it shouldn't be smooth.)

Example 59. Describe the surface S parametrized by

$$\vec{X}(s, t) = (s\cos(t), s\sin(t), s^2), \quad 0 \leq s \leq 2, \ 0 \leq t \leq \pi.$$

By inspection we see that $(x(s, t))^2 + (y(s, t))^2 = s^2 = z(s, t)$, so that S is a portion of the elliptic paraboloid $z = x^2 + y^2$.

Moreover, this looks suspiciously like polar coordinates in the x and y variables (with s playing the role of r, and t playing the role of θ). We therefore see that this is the portion of the paraboloid $z = x^2 + y^2$ with $y \geq 0$ and $z \leq 4$:

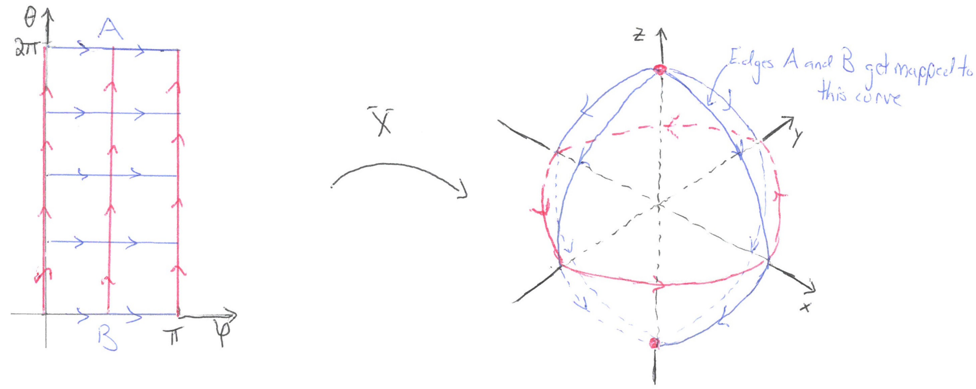


Note that we need *two* parameters s and t to be present in order to parametrize a surface. With only one parameter, we would just get a curve!

Example 60. Let's parametrize the sphere $S: \ x^2 + y^2 + z^2 = 4$.

There are many ways to parametrize a sphere, but perhaps a parametrization inspired by spherical coordinates is the most straightforward. Note that S is described in spherical coordinates using $\rho = 2$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$, so we can parametrize S with

$$\vec{X}(\phi, \theta) = (2\cos(\theta)\sin(\phi), 2\sin(\theta)\sin(\phi), 2\cos(\phi)), \quad 0 \leq \phi \leq \pi, \ 0 \leq \theta \leq 2\pi.$$

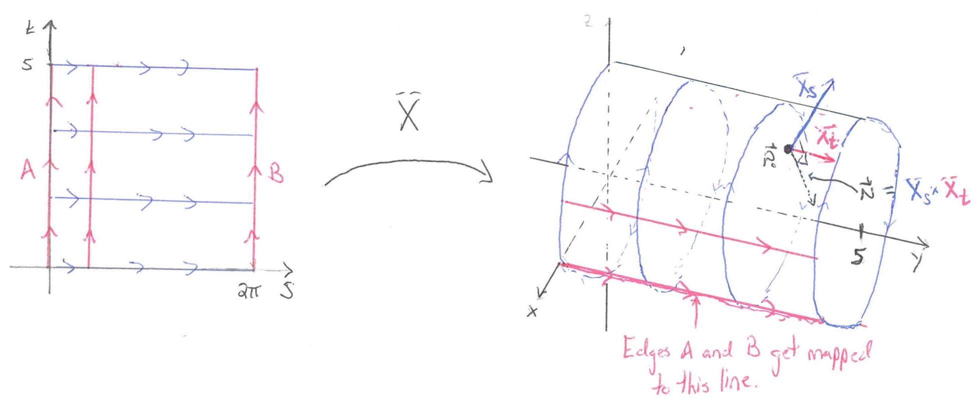


Note that $\vec{X}$ is injective except on the boundary of the box $[0, \pi] \times [0, 2\pi]$, with the left edge of the box being sent to the north pole $(0, 0, 1)$, the right edge of the box being sent to the south pole $(0, 0, -1)$, and the top and bottom of the box both being sent to the portion of the sphere in the xz-plane where $x \geq 0$.

Example 61. Parametrize the portion of the cylinder C (whose axis of symmetry is the y-axis and whose radius is 3) between the xy-plane and the plane $y = 5$.

Here we can describe the cylinder with the inequalities $0 \leq y \leq 5$ and $x^2 + z^2 = 3^2$, so we use an alteration of cylindrical coordinates (with y as the 'vertical' direction and the xz-plane as the 'polar plane') to get

$$\vec{X}(s, t) = (3\cos(s), t, 3\sin(s)), \quad 0 \leq s \leq 2\pi, \; 0 \leq t \leq 5.$$

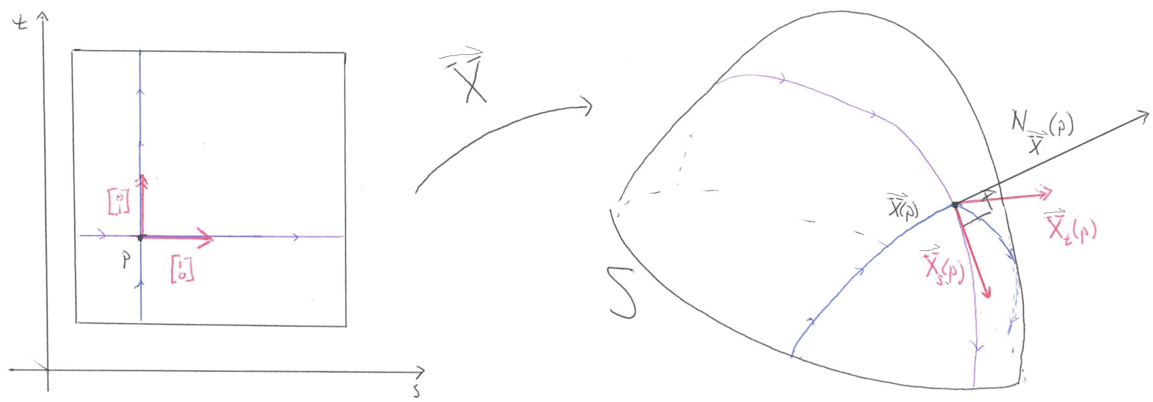


Normal Vectors

We will shortly define what it means for a surface S to be smooth. The "real" definition of smoothness is a little removed from the geometric understanding we have built up for $\mathbb{R}^3$, but there is an equivalent definition that is accessible to us. This can be framed in terms of normal vectors to the surface S at a point. Normal vectors came up last quarter when we discussed the gradient: if $f : \mathbb{R}^3 \to \mathbb{R}$ if C^1 and satisfies $\nabla f(\vec{a}) \neq \vec{0}$, then the level set $S = \{\vec{x} \;:\; f(\vec{x}) = f(\vec{a})\}$ is actually a surface (at least the portion of S near $\vec{a}$), and the vector $\nabla f(\vec{a})$ is normal (i.e. orthogonal, or perpendicular) to S at $\vec{a}$. When we have a parametrization of a surface, the parametrization itselfs gives us a way to find a normal vector to the surface. Let's discuss this now.

Remark 25. Let $\vec{X} : D \to \mathbb{R}^3$ be a C^1 parametrization of a surface S, then fix $\vec{p} = (s_0, t_0) \in D$.

Let's take a moment to discuss the following diagram:



We are interested in studying the surface S at $\vec{X}(\vec{p})$. The first thing that we will do is to get two vectors in $\mathbb{R}^3$, based at $\vec{X}(\vec{p})$, which are tangent to S. To motivate this, we think about the vertical and horizontal paths (which we assume have unit speed) in the st-plane that pass through $\vec{p}$ (pictured in blue and purple, respectively) as being mapped by $\vec{X}$ to curves on S which pass through $\vec{X}(\vec{p})$. In the st-plane, the derivative of the horizontal (purple) path is $\vec{e}_1$ (pictured in red with a single arrow), while the derivative of the vertical (blue) path is $\vec{e}_2$ (pictured in red with a double arrow). The derivative

$$D\vec{X}(p) = \begin{bmatrix} \vec{X}_s(\vec{p}) & \vec{X}_t(\vec{p}) \end{bmatrix}$$

sends the vectors $\vec{e}_1$ and $\vec{e}_2$ based at $\vec{p} \in D$ to the corresponding vectors

$$\vec{X}_s(\vec{p}) = D\vec{X}(\vec{p})\vec{e}_1 \quad \text{and} \quad \vec{X}_t(\vec{p}) = D\vec{X}(\vec{p})\vec{e}_2$$

based at $\vec{X}(\vec{p}) \in S$, shown on S with single- and double-arrows, respectively. Since the purple and blue curves are completely contained in S, the vectors $\vec{X}_s(\vec{p})$ and $\vec{X}_t(\vec{p})$ are tangent to S at $\vec{X}(\vec{p})$.

As long as $\vec{X}_s(\vec{p}), \vec{X}_t(\vec{p})$ form a linearly independent set, we can obtain a normal vector $N_{\vec{X}}(\vec{p})$ to S at $\vec{X}(\vec{p})$ by simply computing the cross product[14] of $\vec{X}_s(\vec{p})$ and $\vec{X}_t(\vec{p})$:

$$N_{\vec{X}}(\vec{p}) \overset{def}{=} \vec{X}_s(\vec{p}) \times \vec{X}_t(\vec{p}) = \begin{bmatrix} y_s(\vec{p})z_t(\vec{p}) - z_s(\vec{p})y_t(\vec{p}) \\ z_s(\vec{p})x_t(\vec{p}) - x_s(\vec{p})z_t(\vec{p}) \\ x_s(\vec{p})y_t(\vec{p}) - y_s(\vec{p})x_t(\vec{p}) \end{bmatrix}.$$

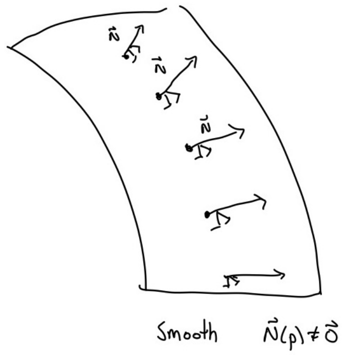
The vector $N_{\vec{X}}(\vec{p})$ is called the **normal vector to** S **at** $\vec{X}(\vec{p})$ **arising from the parametrization** $\vec{X}$. Note that $N_{\vec{X}}(\vec{p})$ is orthogonal to both $\vec{X}_s(\vec{p})$ and $\vec{X}_t(\vec{p})$ (which form a basis for the space of vectors tangent to S at $\vec{X}(\vec{p})$), so that $N_{\vec{X}}(\vec{p})$ is indeed normal to S at $\vec{X}(\vec{p})$.

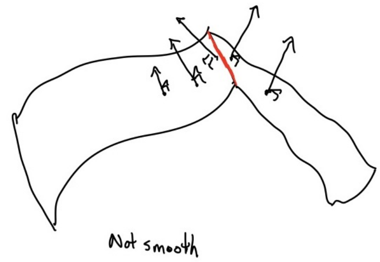
The uses of this normal vector are many. Here are some observations:

(i) We say that that S is **smooth** at a point ☺ $= \vec{X}(\vec{p})$ if $N_{\vec{X}}(\vec{p}) \neq \vec{0}$. The rigorous motivation for this definition involves more differential geometry than we have at our disposal, but for intuition you can think that since $N_{\vec{X}}(\vec{p}) = \vec{X}_s(\vec{p}) \times \vec{X}_t(\vec{p})$ is a continuous function of $\vec{p}$, then if $N_{\vec{X}}(\vec{x}) \neq \vec{0}$

[14] Here is where we are really using the fact that we are working in $\mathbb{R}^3$, as this construction does not work in higher dimensions.

at $\vec{x} = \vec{p}$ then $N_{\vec{X}}(\vec{x}) \neq \vec{0}$ for $\vec{x}$ near $\vec{p}$, and therefore we can make a continuous assignment of non-zero normal vectors to S near ☺. If S failed to be smooth at ☺ (perhaps if it had a corner, or a crease, or some other "non-smooth" feature), then this should result in any assignment of non-zero normal vectors to be discontinuous at $\vec{p}$.



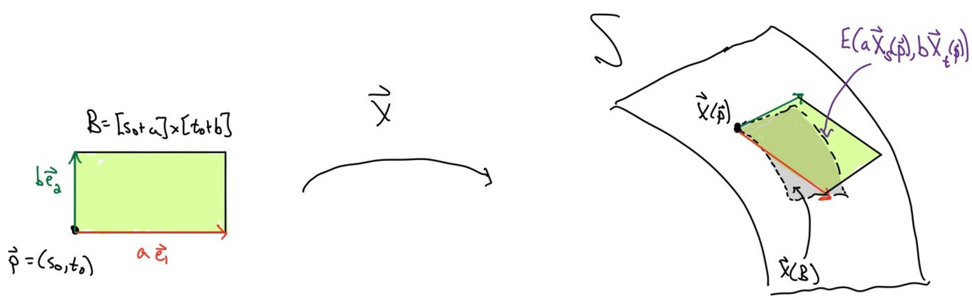


(ii) $\|N_{\vec{X}}(\vec{p})\| = \|\vec{X}_s(\vec{p}) \times \vec{X}_t(\vec{p})\|$ gives the area of the parallelogram determined by $\vec{X}_s(\vec{p})$ and $\vec{X}_t(\vec{p})$ (as shown in Exercise 3 of Homework 3 from MATH 291-2). In particular, for small $a, b > 0$ we have that

$$\vec{X}(\vec{s}) \approx \vec{X}(\vec{p}) + D\vec{X}(\vec{p})(\vec{s} - \vec{p}) \quad \text{for each } \vec{s} \in [s_0 + a] \times [t_0 + b],$$

so that since the image of the box $B = [s_0 + a] \times [t_0 + b]$ under this affine approximation of $\vec{X}$ is the (translated by $\vec{X}(\vec{p})$) parallelogram determined by the vectors $a\vec{X}_s(\vec{p})$ and $b\vec{X}_t(\vec{p})$, it should be that we can reasonable approximate the area of the patch of S given by $\vec{X}(B)$ by

$$\begin{aligned} \operatorname{Vol}_2(E(a\vec{X}_s(\vec{p}), b\vec{X}_t(\vec{p})) &= \|(a\vec{X}_s(\vec{p})) \times (b\vec{X}_t(\vec{p}))\| \\ &= |ab| \|\vec{X}_s(\vec{p}) \times \vec{X}_t(\vec{p})\| \\ &= \|\vec{X}_s(\vec{p}) \times \vec{X}_t(\vec{p})\| \operatorname{Vol}_2(B). \end{aligned}$$



This observation forms the basis for our notion of integration over surfaces (which we will discuss shortly).

Definition 14. A surface $S \subset \mathbb{R}^3$ is called **smooth** if there is a C^1 parametrization $\vec{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ of S such that $N_{\vec{X}}(s, t) \neq \vec{0}$ except possibly on ∂D or on (at most) finitely many other points in D.

Remark 26. Our definitions of smooth curve and smooth surface are convenient for us and that will capture a wide variety of examples, but know that you may see slightly different (and non-equivalent) definitions in a higher-level course on differential geometry.

Example 62. In the cylinder example, we had $\vec{X}_t(s,t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{X}_s(s,t) = \begin{bmatrix} -3\sin(t) \\ 0 \\ 3\cos(t) \end{bmatrix}$, so that

$$N_{\vec{X}}(s,t) = \begin{bmatrix} -3\sin(t) \\ 0 \\ 3\cos(t) \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3\cos(t) \\ 0 \\ -3\sin(t) \end{bmatrix} = -\vec{X}(s,t),$$

so that $N_{\vec{X}}(s,t)$ points "inward" (towards the y-axis) (black in the figure in that example). Note that since $N_{\vec{X}}(s,t) \neq \vec{0}$ for each (s,t) in the domain of $\vec{X}$, the cylinder is a smooth surface.

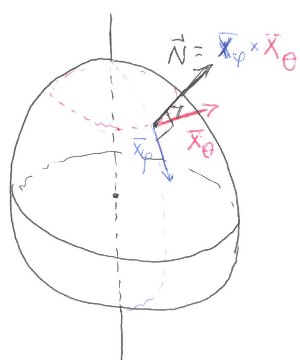
Example 63. For the sphere example, we have

$$\vec{X}_\phi(\phi,\theta) = \begin{bmatrix} 2\cos(\theta)\cos(\phi) \\ 2\sin(\theta)\cos(\phi) \\ -2\sin(\phi) \end{bmatrix} \qquad \text{and} \qquad \vec{X}_\theta(\phi,\theta) = \begin{bmatrix} -2\sin(\theta)\sin(\phi) \\ 2\cos(\theta)\sin(\phi) \\ 0 \end{bmatrix},$$

so that

$$N_{\vec{X}}(\phi,\theta) = \vec{X}_\phi(\phi,\theta) \times \vec{X}_\theta(\phi,\theta) = \begin{bmatrix} 4\cos(\theta)\sin^2(\phi) \\ 4\sin(\theta)\sin^2(\phi) \\ 4\cos(\phi)\sin(\phi) \end{bmatrix} = 2\sin(\phi)\vec{X}(\phi,\theta).$$

Since $2\sin(\phi) > 0$ (well, $= 0$ at the north and south poles), we see that $N_{\vec{X}}(\phi,\theta)$ is a positive multiple of $\vec{X}(\phi,\theta)$, which for the sphere means that it is points "outward" (away from the origin). Note that since $N_{\vec{X}}(\phi,\theta) \neq \vec{0}$ except on the boundary of the domain of $\vec{X}$, the sphere is a smooth surface.



Surface Area and Scalar Surface Integrals

By generalizing our Riemann sum argument that motivated the definition of arclength and scalar line integrals, we are lead to the following definitions.

Definition 15. Let S be a surface with C^1 parametrization $\vec{X} : D \to \mathbb{R}^3$ such that $N_{\vec{X}}(\vec{p}) \neq \vec{0}$ at all $\vec{p} \in D$ (except possibley on ∂D or at finitely many other points). Then we define the surface area of S to be

$$\text{Surface area of } S \overset{def}{=} \iint_D \|N_{\vec{X}}(s,t)\| \, dA(s,t).$$

If $f : S \to \mathbb{R}$ is continuous, then we define the **scalar surface integral** of f over S to be

$$\iint_S f(\vec{x}) \, dS \overset{def}{=} \iint_D f(\vec{X}(s,t)) \|N_{\vec{X}}(s,t)\| \, dA(s,t).$$

Here the notation "dS" is intended to signify an "infinitesimal change in surface area" (just as dA represented an "infinitesimal change in area"). By giving an argument similar to those in the notes (and homework) that arc length and scalar line integrals are well-defined (in the sense that they do not depend on the parametrization one uses), one can show that surface area and scalar surface integrals are also well-defined.

Example 64. Let's compute the surface area of the sphere S_R centered at $(0,0,0)$ of radius R.

We need to compute $\iint_{S_R} 1 dS$. To do this, we first parametrize the sphere with

$$\vec{X}(\phi,\theta) = (R\cos(\theta)\sin(\phi), R\sin(\theta)\cos(\phi), R\cos(\phi)), \quad 0 \leq \pi \leq \phi,\ 0 \leq \theta \leq 2\pi.$$

We need to find $\|N_{\vec{X}}(\phi,\theta)\|$ in order to compute the integral. First, note that

$$\vec{X}_\phi(\phi,\theta) = \begin{bmatrix} R\cos(\theta)\cos(\phi) \\ R\sin(\theta)\cos(\phi) \\ -R\sin(\phi) \end{bmatrix}, \qquad \vec{X}_\theta(\phi,\theta) = \begin{bmatrix} -R\sin(\theta)\sin(\phi) \\ R\cos(\theta)\sin(\phi) \\ 0 \end{bmatrix},$$

so that

$$N_{\vec{X}}(\phi,\theta) = \vec{X}_\phi(\phi,\theta) \times \vec{X}_\theta(\phi,\theta) = \begin{bmatrix} R^2\cos(\theta)\sin^2(\phi) \\ R^2\sin(\theta)\sin^2(\phi) \\ R^2\cos(\theta)\sin(\phi) \end{bmatrix} = R^2\sin(\phi) \begin{bmatrix} \cos(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\phi) \end{bmatrix},$$

and hence

$$\|N_{\vec{X}}(\phi,\theta)\| = R^2\sin(\phi)\sqrt{\cos^2(\theta)\sin^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\phi)} = R^2\sin(\phi).$$

We therefore have

$$\iint_S 1 dS = \int_0^{2\pi}\int_0^{\pi} 1\|N_{\vec{X}}(\phi,\theta)\| d\phi d\theta = \int_0^{2\pi}\int_0^{\pi} R^2\sin(\phi) d\phi d\theta = 4\pi R^2,$$

which is the well-known formula for the surface-area of a sphere of radius R!

Example 65. Let's compute the surface area of the right circular cone C of top radius R and height H, given by $z = \frac{H}{R}\sqrt{x^2+y^2}$, $0 < z \leq H$.

It might be best to parametrize the cone using cylindrical coordinates. That is, we write

$$\vec{X}(r,\theta) = (r\cos(\theta), r\sin(\theta), \frac{H}{R}r), \quad 0 < r \leq R, \quad 0 \leq \theta \leq 2\pi.$$

Then we have

$$\vec{X}_r(r,\theta) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \frac{H}{R} \end{bmatrix}, \qquad \vec{X}_\theta(r,\theta) = \begin{bmatrix} -r\sin(\theta) \\ r\cos(\theta) \\ 0 \end{bmatrix},$$

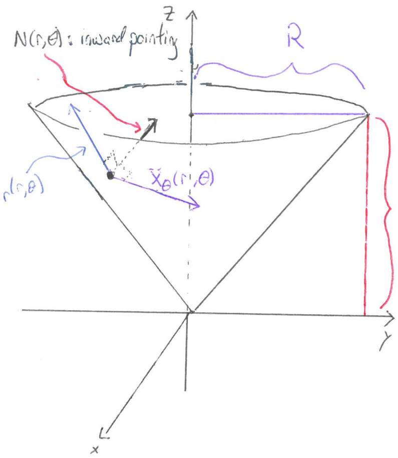
so that

$$N_{\vec{X}}(r,\theta) = \vec{X}_r(r,\theta) \times \vec{X}_\theta(r,\theta) = \begin{bmatrix} -\frac{H}{R}r\cos(\theta) \\ -\frac{H}{R}r\sin(\theta) \\ r \end{bmatrix},$$

and therefore

$$\|N_{\vec{X}}(r,\theta)\| = r\sqrt{\left(\frac{H}{R}\right)^2 + 1} = \frac{r}{R}\sqrt{H^2+R^2}.$$

Note that for $r > 0$ (i.e. for every point on C except for $(0,0,0)$), the $\vec{k}$-component of $N_{\vec{X}}$ is positive, so that $N_{\vec{X}}$ must be the 'inward pointing' normal vector:



We therefore have

$$\text{Sur. Area of } C = \iint_C 1 dS = \int_0^{2\pi}\int_0^R 1\|\vec{N}(r,\theta)\| dr d\theta = \frac{\sqrt{H^2+R^2}}{R}\int_0^{2\pi}\int_0^R r dr d\theta = \pi R\sqrt{H^2+R^2}.$$

Lecture 16: Vector Fields

Learning Objectives:

- Visualize vector fields.
- Analyze the flow lines of a vector fields.
- Interpret the gradient of a function as a vector field.

Today we introduce a class of object that, one might argue, vector calculus was designed to study: vector fields. Vector fields provide a physical interpretation for *differential forms*, which are the natural objects that link differentiation with integration.

Definition 16. Let $\Omega \subseteq \mathbb{R}^n$. A function $\vec{F} : \Omega \to \mathbb{R}^n$ is called a **vector field** on Ω.

In the past we have thought about such functions as maps from (a subset of) $\mathbb{R}^n$ to $\mathbb{R}^n$, and therefore as some sort of transformation of space. Now, however, we shift gears and think about such maps as assigning to each point $\vec{x} \in \mathbb{R}^n$ a vector $\vec{F}(\vec{x}) \in \mathbb{R}^n$. That is, we think of $\vec{x}$ as being a point in space, and $\vec{F}(\vec{x})$ as a vector based at $\vec{x}$.

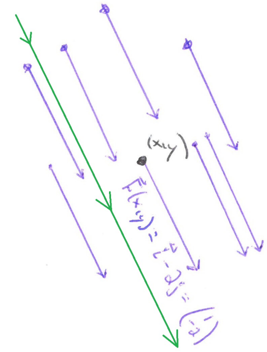
The physical interpretation is that $\vec{F}(\vec{x})$ is a **force** acting at the point $\vec{x}$, and therefore $\vec{F}$ describes a force field that governs the flow of particles around space.

For example, if we think of space as a fluid, then $\vec{F}(\vec{x})$ describes the direction and magnitude of the flow of the fluid at the point $\vec{x}$. If $\vec{x}(t)$ represents the path taken by some particle that is carried along by the current, then we expect the velocity $\vec{x}\,'(t)$ of the particle to be determined by the force $\vec{F}$, in the sense that $\vec{x}\,'(t) = \vec{F}(\vec{x}(t))$.

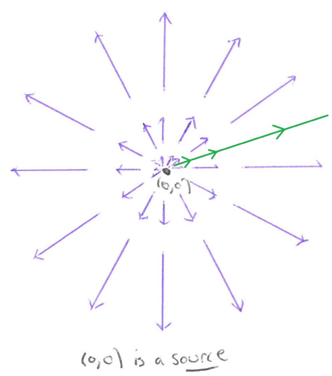
Definition 17. Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and let $I \subseteq \mathbb{R}$ be an interval. A differentiable path $\vec{x} : I \to \mathbb{R}^n$ satisfying $\vec{x}\,'(t) = \vec{F}(\vec{x}(t))$ at each $t \in I$ is called a **flow line** of $\vec{F}$.

Besides fluid flow, we also think of gravity as generating such a force field. There are other physical interpretations as well, but these are the ones with which you are likely to already be familiar.

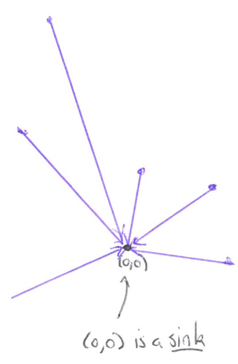
Example 66. The vector field $\vec{F}(x, y) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a constant field. If this were the velocity field of some fluid then it would describe a **uniform flow**, where all of the fluid flows in the same direction at the same speed. A flow line is drawn in green. Indeed, the flow line through (a, b) with $\vec{x}(0) = (a, b)$ is $\vec{x}(t) = (a + t, b - 2t)$, since $\vec{x}(0) = (a, b)$ and $\vec{x}\,'(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{F}(\vec{x}(t))$.



Example 67. On the other hand, the field $\vec{F}(x,y) = x\vec{i} + y\vec{j}$ is pushes all points directly away from the origin at a speed proportional to the distance from the origin. Since points near the origin are pushed away from the origin, the origin is called a **source** for this flow. A flow line is drawn in green; note that a particle on the flow line moves faster as it gets further away from the origin (since $\vec{F}$ is larger in magnitude at points further away from the origin). Indeed, the flow line $\vec{x}$ that satisfies $\vec{x}(0) = (a,b)$ is $\vec{x}(t) = (ae^t, be^t)$. To verify this, note that $\vec{x}(0) = (ae^0, be^0) = (a,b)$ and $\vec{x}\,'(t) = \begin{bmatrix} ae^t \\ be^t \end{bmatrix} = \vec{F}(ae^t, be^t) = \vec{F}(\vec{x}(t))$.



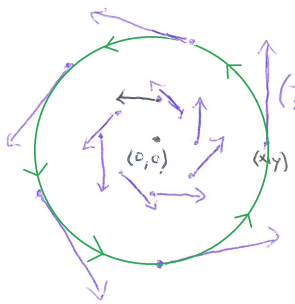
If instead we looked at $\vec{F}(x,y) = -x\vec{i} - y\vec{j}$, then all points are getting pulled towards the origin at a speed proportional to the distance from the origin. Since all points near the origin are pulled towards the origin, the origin is called a **sink** for this flow.

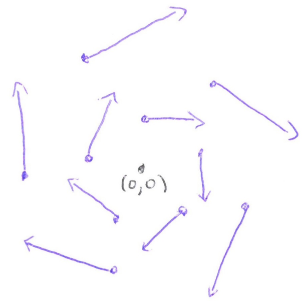


Example 68. Besides transporting, compressing, and expanding, fluids can also rotate. Indeed, the field $\vec{F}(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix}$ (below left) describes a flow which rotates space in a counter-clockwise fashion around the origin, and the speed at which they are rotated is proportional to their distance to the origin.

We could also have $\vec{F}(x,y) = \begin{bmatrix} y \\ -x \end{bmatrix}$ (below right), which is the same as the previous flow but in the clockwise direction. A flow line for the counterclockwise flow in sketched in green. Note that every path of the form $\vec{x}(t) = (R\cos(t), R\sin(t))$ (for $R > 0$) is a flow line for $\vec{F}$, since

$$\vec{x}\,'(t) = \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix} = \vec{F}(R\cos(t), R\sin(t)) = \vec{F}(\vec{x}(t)).$$

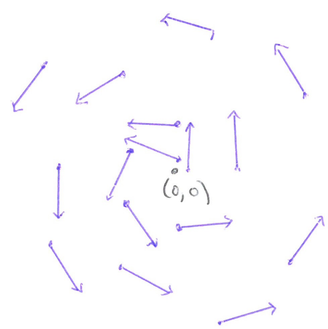




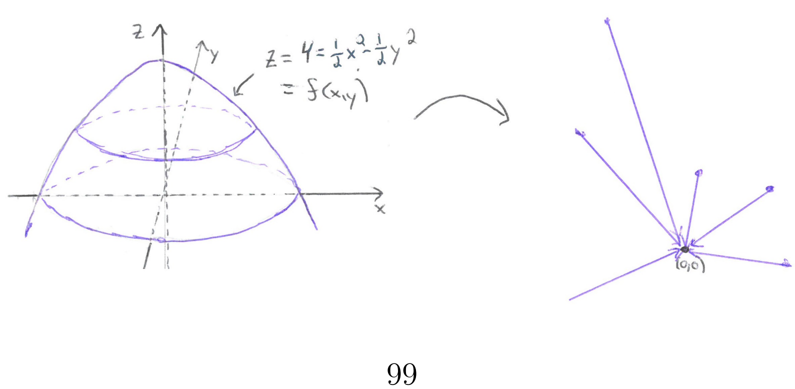
Not every vector field has to be nice. For example, we can look at

$$\vec{F}(x,y) = \frac{-y}{\sqrt{x^2+y^2}}\vec{i} + \frac{x}{\sqrt{x^2+y^2}}\vec{j},$$

which rotates point around the origin in a counterclockwise fashion, but always at speed 1. Thus, points very close to the origin are rapidly spinning around the origin, while points far away from the origin are revolving quite slowly.



Example 69. Technically, the gradient of a function is also a vector field. Indeed, the vector field $\vec{F}(x,y) = \begin{bmatrix} -x \\ -y \end{bmatrix}$ is secretly $\vec{F}(x,y) = \nabla f(x,y)$, where $f(x,y) = 4 - \frac{1}{2}x^2 - \frac{1}{2}y^2$ is the downward-opening elliptic paraboloid. Thus, we can think of $\vec{F}(x,y)$ as describing the (x,y)-direction in which a hiker on the graph of $z = f(x,y)$ should walk in order to climb uphill as quickly as possible.



Vector fields of the type in the previous example (i.e. that are the gradients of scalar-valued functions) are very important in calculus, and we will given them a name.

Definition 18. Let $\Omega \subseteq \mathbb{R}^n$, and let $\vec{F} : \Omega \to \mathbb{R}^n$ be a vector field on Ω. If $\vec{F} = \nabla f$ for some C^1 function $f : \Omega \to \mathbb{R}$, then we call $\vec{F}$ a **gradient field** (or **conservative field**) with **potential function** f.

Example 70. Not every vector field is a gradient field. For one counterexample, consider

$$\vec{F}(x,y,z) = \begin{bmatrix} xy^2 \\ x^2y \\ xy \end{bmatrix}.$$

To see why, we note that if $\vec{F} = \nabla f$, then $f_x(x,y,z) = xy^2$ and $f_z(x,y,z) = xy$, but by Clairaut's theorem we would have $0 = (f_x)_z(x,y,z) = (f_z)_x(x,y,z) = y$, which fails to hold (for example) at $(0,1,0)$.

Example 71. One important (and perhaps) surprising point is that a vector field $\vec{F} : \Omega \to \mathbb{R}^n$ may have a potential function on a ball centered at each point in Ω, but may not actually have a potential function on Ω.

The classical example of such a vector field is

$$\vec{F} : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2, \quad \vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}.$$

We will prove later in the course that there is no function $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ such that $\nabla f(x,y) = \vec{F}(x,y)$ throughout $\mathbb{R}^2 - \{(0,0)\}$. On the other hand, note that

$$f : \{(x,y) \ : \ x > 0\} \to \mathbb{R}, \ f(x,y) = \arctan\left(\frac{y}{x}\right)$$

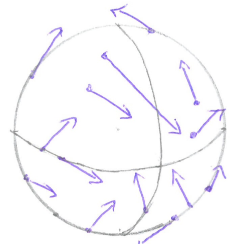
satisfies $\nabla f(x,y) = \vec{F}(x,y)$ at each $(x,y) \in \{(x,y) \ : \ x > 0\}$, and therefore f is a potential function on this set. In one of your homework problems this week, you will argue that $\vec{F}$ actually has potential functions on the other open-half planes as well. The counterintuitive fact is that we just cannot "glue" these potentials together to produce a potential function for $\vec{F}$ that is valid throughout $\mathbb{R}^2 - \{(0,0)\}$.

For a somewhat deep explanation for why this is, one can show that if $\theta(x,y)$ is a (continuous) choice of polar angle for (x,y) in a region in $\mathbb{R}^2 - \{(0,0)\}$ then one can show that $\nabla\theta(x,y) = \vec{F}(x,y)$. Although $\theta(x,y)$ might seem like a potential function for $\vec{F}$ on $\mathbb{R}^2 - \{(0,0)\}$, the fact is that we cannot define $\theta(x,y)$ continuously throughout $\mathbb{R}^2 - \{(0,0)\}$ all at once, since if we start at a point (x,y) and move along a path that ends and (x,y) and rotates once around the origin in the counterclockwise direction, it will be that θ will increase by 2π along this path. This observation actually plays a *huge* role in complex analysis, where it is used to define (for example) various different versions of inverse functions (like logarithms, roots, inverse trigonometric functions, etc.) to suit our purposes.

As an application of some of these ideas, let's discuss a fun theorem involving vector fields: the Hairy Ball Theorem.

Hairy Ball Theorem

Suppose that the surface of the earth (which, for our purposes, is the sphere $x^2 + y^2 + z^2 = 1$) is completely covered in water (here, let's make the assumption that the water doesn't behave differently at different depths). Then at every point $\vec{p}$ on the earth the velocity of the water is given by some vector $\vec{F}(\vec{p})$. For physical reasons, we can assume that $\vec{F}$ is continuous.



Now, suppose a particle (a life-raft?) is floating along on the surface of the water, and its movements are governed by $\vec{F}$. The life-raft travels along some flow line $\vec{r}(t)$ of the vector field. Note that, since the boat is on the surface of the earth (and therefore $\|\vec{r}(t)\| = 1$ at every time t), its velocity $\vec{r}\,'(t)$ is always perpendicular to its position vector $\vec{r}(t)$. (Indeed, this is one of your homework problems for this week!) That is, its velocity vector is always tangent to the surface of the earth. In particular, the vector $\vec{F}(\vec{p})$ at $\vec{p}$ is **tangent** to the sphere at $\vec{p}$.

One (perhaps surprising) fact is that, in the scenario that we described, *there will always be a point on the sphere at which* $\vec{F}(\vec{p}) = \vec{0}$. That is, there is always at least one point on the surface of the earth at which the velocity of the flow is $\vec{0}$. This result is known as the 'Hairy Ball Theorem', since one can think of the flow vectors $\vec{F}(\vec{p})$ as hairs on the sphere which are tangent to the sphere (i.e. they have been 'combed').

Theorem 11 (Hairy Ball)**.** Let $S \subset \mathbb{R}^3$ be a sphere. If $\vec{F}$ is a continuous vector field on S which is tangent to S at each point, then there is some point $\vec{p}$ on S at which $\vec{F}(\vec{p}) = \vec{0}$.

The hairy ball theorem implies that, if you have a hairy ball such as this one, where the direction and length of the hairs are continuous and the hairs are forced to lie flat against the ball, then there must be a bald spot. [15]

[15] There are a ton of fun theorems like this in higher mathematics. For a course that's loaded with them, you should take... Algebraic Topology! (Surprised I didn't say Real or Complex Analysis?)

Lecture 17: Gradient, Divergence, and Curl

Learning Objectives:

- Compute the divergence of a vector field.
- Describe the geometric meaning of divergence.
- Compute the curl of a two- or three-dimensional vector field.
- Describe the geometric meaning of curl.
- Investigate the relationship between grad, curl, and div.
- Apply Poincaré's Lemma to determine when a vector field is conservative over a set.

Today we introduce various notions of the derivative related to vector fields. These are all special cases of (what we will call) the *exterior derivative* for differential forms, but historically they arose from physical considerations. To this end, we will adopt some language from fluid dynamics to describe certain ideas related to these derivatives.

The Del Operator: Grad, Div, and Curl

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the notions of derivative we consider can be captured (informally) described in terms of the **del** operator:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}.$$

To be clear, this is purely notation that will help us remember formulas. However, it does come in handy for describing various types of the derivative. For example, we can write the gradient of a function $f(x, y, z)$ as

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f(x, y, z) = \begin{bmatrix} f_x(x, y, z) \\ f_y(x, y, z) \\ f_z(x, y, z) \end{bmatrix}.$$

We will sometimes also write $\text{grad}(f)$ for ∇f.

Divergence and Incompressibility

Although it only makes sense to take the gradient of functions, we can use the del operator to define various notions of the derivative for vector fields.

Definition 19. If $\Omega \subseteq \mathbb{R}^n$ and $\vec{F} = (F_1, \dots, F_n) : \Omega \to \mathbb{R}^n$ is a C^1 vector field on Ω, then we define the **divergence** of $\vec{F}$ to be

$$\text{div}\vec{F}(\vec{x}) = \nabla \cdot \vec{F}(\vec{x}) \overset{def}{=} \frac{\partial F_1}{\partial x_1}(\vec{x}) + \cdots + \frac{\partial F_n}{\partial x_n}(\vec{x}).$$

We will see that physically, if $\vec{F}$ describes the motion of some two- or three-dimensional fluid, the number $\text{div}\vec{F}(\vec{x})$ measures the net amount of expansion of the fluid at $(\vec{x})$, with positive values indicating that the fluid is expanding a $\vec{x}$, and negative values indicating that the fluid is compressing at $\vec{x}$. If $\text{div}\vec{F}(\vec{x}) = 0$ at every point $\vec{x} \in \Omega$, then $\vec{F}$ is called **incompressible**.

Although it may be unclear why this is the intuition for divergence, it will become clear once we learn the generalization of the Fundamental Theorem of Calculus that involves divergence (called the Divergence Theorem or Gauss's Theorem).

Example 72. Consider the uniform flow $\vec{F}(x, y) = \vec{i} - 2\vec{j}$. Intuitively, the fluid described by $\vec{F}$ is neither expanding nor contracting at any point. Thus, we expect $\text{div}\vec{F}$ to be 0 at every point. Indeed, we have $\text{div}\vec{F}(x, y) = (1)_x + (-2)_y = 0$ at every point, as expected.

Similarly, the flow $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$ corresponding to counterclockwise rotation of the plane should also have $\text{div}\vec{F}(x, y) = 0$ at every point, since this flow corresponds to rigidly rotating the plane about the origin (so that there is no expansion or contraction anywhere). Indeed, we can compute that $\text{div}\vec{F}(x, y) = (-y)_x + (x)_y = 0$, as expected.

On the other hand, consider the flow $\vec{F}(x, y) = x\vec{i} + y\vec{j}$ (for which the origin was a source). This flow pushes all of the points away from the origin, and in the process the fluid is stretching apart at each point. Thus, we expect this vector field to have positive divergence at every point. Indeed, $\text{div}\vec{F}(x, y) = (x)_x + (y)_y = 2$, which agrees with our intuition. On the other hand, the flow $\vec{F}(x, y) = -x\vec{i} - y\vec{j}$ (for which the origin is a sink) has $\text{div}\vec{F}(x, y) = -2$, which agrees with our intuition that this field compresses the fluid at each point (as it sends the fluid towards the origin).

Curl and Irrotationality

We can also utilize the cross-product to generate a notion of the derivative for vector fields (this time in $\mathbb{R}^3$ only).

Definition 20. If $\Omega \subseteq \mathbb{R}^3$ and if $\vec{F} = (P, Q, R) : \Omega \to \mathbb{R}^3$ is a C^1 vector field on Ω, then we define the **curl** of $\vec{F}$ to be

$$\text{curl}\vec{F}(x, y, z) = \nabla \times \vec{F}(x, y, z) \stackrel{def}{=} \begin{bmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{bmatrix} = \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\vec{i} + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\vec{j} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\vec{k}.$$

Remark 27. It is sometimes helpful to have a notion of curl for two-dimensional vector fields as well. If $\Omega \subseteq \mathbb{R}^2$ and if $\vec{F} = (P, Q) : \Omega \to \mathbb{R}^2$ is a C^1 vector field on Ω, then we define the **scalar curl** of $\vec{F}$ to be

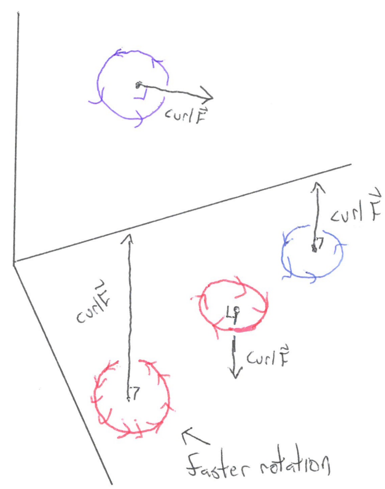
$$\text{curl}\vec{F}(x, y) = \frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y).$$

The scalar curl of a two-dimensional vector field $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ can be seen as coming from the curl of $\tilde{\vec{F}}(x,y,z) = P(x,y)\vec{i} + Q(x,y)\vec{j} + 0\vec{k}$, in the sense that

$$\begin{aligned}
\text{curl}\vec{F}(x,y) &= \frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) \\
&= \vec{k} \cdot \Big(0\vec{i} + 0\vec{j} + \Big(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\Big)\vec{k}\Big) \\
&= \vec{k} \cdot \text{curl}\tilde{\vec{F}}(x,y,0),
\end{aligned}$$

since $\frac{\partial 0}{\partial x} = 0 = \frac{\partial 0}{\partial y}$.

We will see that physically, curl$\vec{F}$ captures how the fluid described by $\vec{F}$ is rotating (in the sense of 'twisting') at a point. The information here is encoded in a very geometric way: for a three-dimensional vector field $\vec{F}$, curl$\vec{F}(x,y,z)$ is normal to the plane of rotation of the fluid at (x,y,z), and the length of curl$\vec{F}(x,y,z)$ determines the angular speed of the rotation. The direction of rotation is determined by the direction of curl$\vec{F}(x,y,z)$ and the right-hand-rule. That is, if the heel of your right hand sits of the plane of rotation, and your fingers curl in the direction of the rotation, then your thumb will point in the same direction as curl$\vec{F}(x,y,z)$.



For two-dimensional vector fields $\vec{F}$, curl$\vec{F}(x,y)$ is a scalar quantity. If curl$\vec{F}(x,y) > 0$, then the rotation of $\vec{F}$ at (x,y) is counterclockwise (in the xy-plane). If curl$\vec{F}(x,y) < 0$, then the rotation of $\vec{F}$ at (x,y) is clockwise. Note that if $\tilde{\vec{F}}$ is the "extension" of $\vec{F}$ to three dimensions mentioned in Remark 27, then curl$\vec{F}(x,y) = \vec{k} \cdot \text{curl}\tilde{\vec{F}}(x,y,0)$. But the $\vec{i}$ and $\vec{j}$ components of curl$\tilde{\vec{F}}(x,y,0)$ are zero (as noted above), so that $\tilde{\vec{F}}(x,y,0)$ is already normal to the xy-plane (which must be the plane in which any rotation of $\vec{F}$ is happening at all, since $\vec{F}$ is a two-dimensional vector field on the xy-plane). Therefore the $\vec{k}$ component of $\tilde{F}(x,y,0)$—namely curl$\vec{F}(x,y)$—is positive when the rotation of $\tilde{\vec{F}}$ at $(x,y,0)$ is counterclockwise when viewed from the positive z-direction, and is negative when the rotation of $\tilde{\vec{F}}$ at $(x,y,0)$ is clockwise when viewed from the positive z-direction.

If the fluid is not twisting at a point (x,y,z), then curl$\vec{F}(x,y,z) = \vec{0}$. If curl$\vec{F}(x,y,z) = \vec{0}$ at *every point*, then the fluid described by $\vec{F}$ is called **irrotational**.

Again, we currently have no justification for this intuition. However, we will justify it later when we talk about two generalizations of the Fundamental Theorem of Calculus: Green's Theorem and Stoke's Theorem.

Example 73. The constant flow $\vec{F}(x,y,z) = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$ does not exhibit any rotation or twisting at any point, so we expect it to be irrotational. Indeed, we can easily compute that $\text{curl}\vec{F}(x,y,z) = \vec{0}$ for every point (x,y,z).

On the other hand, the flow $\vec{F}(x,y,z) = -y\vec{\imath}+x\vec{\jmath}$, which describes rotation around the z-axis, satisfies $\text{curl}\vec{F}(x,y,z) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, indicating that there is twisting happening near each point in the xy-plane, and that this twisting is counterclockwise when viewed from the positive z-axis (since $\text{curl}\vec{F}(x,y,z)$ points upwards at every point).

Similarly, $\vec{F}(x,y,z) = z\vec{\jmath} - y\vec{k}$, describing rotation around the x-axis, satisfies $\text{curl}\vec{F}(x,y,z) = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$, indicating that the twisting is counterclockwise when viewed from the *negative* x-axis.

Example 74. To distinguish rotation at a point (i.e. twisting) with large-scale rotation, note that $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{\imath} + \frac{x}{x^2+y^2}\vec{\jmath}$ describes counterclockwise rotation around the origin $\mathbb{R}^2 - \{(0,0)\}$, but you can compute that $\text{curl}\vec{F}(x,y,z) = \vec{0}$ at each point, and therefore the fluid is not twisting at any point (even though it is rotating on the large scale!).

We just looked at simple examples here, but a general vector field may be expanding in some points, contracting in others, and display lots of wild rotation (whirlpools, etc). The study of vector fields in terms of their divergence and curl plays a big role in Fluid Dynamics.

Gradient, Curl, and Divergence as Differential Operators

Note that since the gradient of a function is a vector field, the curl of a vector field is a vector field, and the divergence of a vector field is a function, there is a sensible ordering of our operations in $\mathbb{R}^3$ as

$$\text{(functions)} \xrightarrow{grad} \text{(vector fields)} \xrightarrow{curl} \text{(vector fields)} \xrightarrow{div} \text{(functions)}.$$

There is a curious thing that happens when one composes these operations in this way. For example, if $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 function, then

$$\text{curl}(\text{grad}(f(\vec{x}))) = \text{curl}\begin{bmatrix} f_x(\vec{x}) \\ f_y(\vec{x}) \\ f_z(\vec{x}) \end{bmatrix} = \begin{bmatrix} f_{zy}(\vec{x}) - f_{yz}(\vec{x}) \\ f_{xz}(\vec{x}) - f_{zx}(\vec{x}) \\ f_{yx}(\vec{x}) - f_{xy}(\vec{x}) \end{bmatrix} = \vec{0}$$

by Clairaut's Theorem. Moreover, a similar argument shows that if $\vec{F}(\vec{x}) = \begin{bmatrix} P(\vec{x}) \\ Q(\vec{x}) \\ R(\vec{x}) \end{bmatrix}$ is a C^2 vector field on $\mathbb{R}^3$, then

$$\begin{aligned} \text{div}(\text{curl}(\vec{F}(\vec{x}))) &= \text{div}\begin{bmatrix} R_y(\vec{x}) - Q_z(\vec{x}) \\ P_z(\vec{x}) - R_x(\vec{x}) \\ Q_x(\vec{x}) - P_y(\vec{x}) \end{bmatrix} \\ &= R_{yx}(\vec{x}) - Q_{zx}(\vec{x}) + P_{zy}(\vec{x}) - R_{xy}(\vec{x}) + Q_{xz}(\vec{x}) - P_{yz}(\vec{x}) \\ &= P_{zy}(\vec{x}) - P_{yz}(\vec{x}) + Q_{xz}(\vec{x}) - Q_{zx}(\vec{x}) + R_{yx}(\vec{x}) - R_{xy}(\vec{x}) \\ &= 0, \end{aligned}$$

again by Clairaut's Theorem. Therefore, we have shown that

$$\text{curl}(\text{grad}(f(x,y,z))) = \vec{0} \quad \text{and} \quad \text{div}(\text{curl}(\vec{F})(x,y,z)) = \vec{0}.$$

In the two-dimensional case, we have

$$\text{curl}(\text{grad}(f(x,y))) = \frac{\partial^2 f}{\partial x \partial y}(x,y) - \frac{\partial^2 f}{\partial y \partial x}(x,y) = 0.$$

This is no accident, and (perhaps surprisingly) we will see that this is closely tied to the geometric fact that a (piecewise smooth) curve that is the edge of a smooth surface does not have any endpoints, and that a (piecewise smooth) surface that is the boundary of an elementary region does not have any edges. We will revisit this later in the quarter!

Poincaré's Lemma

From our observations above, if $\Omega \subseteq \mathbb{R}^n$ (for $n = 2,3$) and $\vec{F} : \Omega \to \mathbb{R}^n$ ($n = 2,3$) is conservative (i.e. if $\vec{F} = \nabla f$), then we must have curl$\vec{F}$ = curl(grad(f)) = 0 at each point. One might hope that this test can be used in reverse: that is, one might hope that if curl$\vec{F} = \vec{0}$ on a region D, then $\vec{F}$ is conservative on D. Unfortunately, though, this is not always the case.

Example 75. Recall that the vector field $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\vec{i} + \dfrac{x}{x^2+y^2}\vec{j}$ satisfies curl$\vec{F}(x,y) = 0$ throughout $\mathbb{R}^2 - \{(0,0)\}$, but that (for now taken for granted) there is no function $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ with $\vec{F} = \nabla f$ throughout $\mathbb{R}^2 - \{(0,0)\}$. Nevertheless, as you will prove on your homework, $\vec{F}$ is conservative on smaller subregions in $\mathbb{R}^2 - \{(0,0)\}$. For example, $\vec{F}$ is conservative on the set $\{(x,y) : x > 0\}$, which is the half of the plane to the right of the y-axis (indeed, $\vec{F}(x,y) = \nabla\Big(\arctan\left(\frac{y}{x}\right)\Big)$ on this set).

So why is $\vec{F}$ conservative on some smaller regions in $\mathbb{R}^2 - \{(0,0)\}$, but not on the whole punctured plane? This sort of bad behavior can happen on $\mathbb{R}^2 - \{(0,0)\}$ because this set is not **simply connected**. To say that a region D in $\mathbb{R}^2$ or $\mathbb{R}^3$ is simply-connected means two things: first, the region must be in one piece (i.e. it must be **connected**). Secondly (and this is the meat of the term 'simply connected'), every closed curve (i.e. a curve that forms a "loop", without endpoints) in D can be continuously shrunk (while staying in D!) to a point[16].

In this example, note that the simple closed curve $x^2 + y^2 = 1$ cannot be shrunk to a point in $\mathbb{R}^2 - \{(0,0)\}$ without somehow passing through the missing point $(0,0)$, and therefore $\mathbb{R}^2 - \{(0,0)\}$ is not simply connected. On the other hand, the open half-plane $\{(x,y) : x > 0\}$ is simply connected. In two-dimensions, simple connectedness can be roughly understood as saying that the region doesn't have any "holes" (in the way that $\mathbb{R}^2 - \{(0,0)\}$ has a 'hole' at $(0,0)$). In three-dimensions, the notion is slightly more nuanced (for example, $\mathbb{R}^3 - \{(0,0,0)\}$ is simply connected, but $\mathbb{R}^3 - \{(0,0,z) : z \in \mathbb{R}\}$ is not).

As it turns out, simple connectedness is the magical missing piece that allows us to see that a vector field is conservative[17]:

Theorem 12 (Poincaré's Lemma)**.** If $\vec{F}$ is a C^1 vector field defined in a simply connected region $D \subset \mathbb{R}^n$ ($n = 2,3$), and if curl$\vec{F}$ is zero at each point in D, then $\vec{F}$ is conservative on D.

The proof of this result is well beyond the scope of the course, but you may encounter it in a differential geometry course.

[16]This is an admittedly very hand-wavy definition, but you will learn the rigorous definition of what "continuously shrunk to a point" means (and how to work with it) in MATH 344-2: Introduction to (Algebraic) Topology!

[17]Technically, this is not Poincareé's Lemma, but rather a more difficult result that one can prove using Poincaré's Lemma.

Lecture 18: Differential Forms

Learning Objectives:

- Demonstrate familiarity with the basic definitions and mechanics of differential forms.
- Compute the differential of a function, and the exterior derivative of a differential form.

Our next major task in the course is to investigate various generalizations of the Fundamental Theorem of Calculus, which relates integration to antidifferentiation. This is one of the great stories of modern mathematics, and the story is made much more cohesive when framed in terms of *differential forms*. Differential forms are, in short, the objects that we will integrate going forward. They are computationally efficient, and come with notions of differentiation, integration, and change-of-variables.

To avoid going too far afield, we will treat differential forms as a formal construct, and make definitions based on this construct. However, know that there is a lot more going on here that would be more adequately explored in the course in Differential Geometry. When possible, I will add deeper explanations about certain aspects of differential forms.

0-forms and 1-forms

Definition 21. Let $E \subseteq \mathbb{R}^n$ be open.

- A 0-form on E is a function $f : E \to \mathbb{R}$.
- A 1-form on E is an expression ω of the form

$$\omega = f_1\,dx_1 + \cdots + f_n\,dx_n, \quad \text{where } f_1, \ldots, f_n : E \to \mathbb{R}.$$

  We say that ω is a C^1 (or C^2, or whatever adjective you wish) differential form if $f_1, \ldots, f_n$ are C^1 (or C^2, or whatever).

Example 76. For example, $\omega = -y\,dx + x\,dy$ is a 1-form on $\mathbb{R}^2$. As written, ω could also be viewed as a 1-form on $\mathbb{R}^3$, with $\omega = -y\,dx + x\,dy + 0\,dz$.

Remark 28. When writing out a differential form ω, we will often omit the dx_j term if the coefficient of dx_j is the zero function. Note that this term is still technically "there", but we are just choosing not to write it for the sake of efficiency. For example, we can write $e^x y\,dy$ instead of $0\,dx + e^x y\,dy$ for the sake of brevity.

Remark 29. You should think of the space of 1-forms as the set of all linear combinations of the symbols $dx_1, \ldots, dx_n$, where the coefficients in the linear combination are functions of $\vec{x} = (x_1, \ldots, x_n)$. This looks a lot like a vector space, but is not technically a vector space because the collection of coefficients (the space of 0-forms) is not a *field* because not every 0-form has a multiplicative inverse. Instead, this is a more general construction called a **module**. You do not need to know about modules in order to work with differential forms, but it is helpful to know that the space of 1-forms is similar to a vector space. To hammer this home, consider the following definition.

Definition 22. Given two 1-forms

$$\omega = f_1\,dx_1 + \cdots + f_n\,dx_n \quad \text{and} \quad \eta = g_1\,dx_1 + \cdots + g_n\,dx_n$$

on $E \subseteq \mathbb{R}^n$, we define the 1-form $\omega + \eta$ as

$$\omega + \eta \stackrel{def}{=} (f_1 + g_1)\,dx_1 + \cdots + (f_n + g_n)\,dx_n.$$

If $h : E \to \mathbb{R}$ is a 0-form, then we define the 1-form $h\omega$ as

$$h\omega \stackrel{def}{=} hf_1\,dx_1 + \cdots + hf_n\,dx_n.$$

The Differential of a 0-Form

For each C^1 function there is a very important 1-form associated to the function, called the *differential* of the function, that is the 'differential form' version of the derivative of f.

Definition 23. If $f : E \to \mathbb{R}$ is a C^1 0-form, then the **differential** of f, denoted df, is the 1-form defined by

$$df \stackrel{def}{=} \frac{\partial f}{\partial x_1}\,dx_1 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n.$$

Example 77. For a concrete example, note that if $f(x,y,z) = z^2e^{xy}$ is a 0-form on $\mathbb{R}^3$, then

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = yz^2e^{xy}dx + xz^2e^{xy}dy + 2ze^{xy}\,dz.$$

Remark 30. Note that the differential operator is linear, in the sense that if $f, g : \mathbb{R}^n \to \mathbb{R}$ are two 0-forms and if $k \in \mathbb{R}$, then

$$\begin{aligned} d(f + kg) &= \Big(\frac{\partial f}{\partial x_1} + k\frac{\partial g}{\partial x_1}\Big)dx_1 + \cdots + \Big(\frac{\partial f}{\partial x_n} + k\frac{\partial g}{\partial x_n})dx_n \\ &= \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n + k\Big(\frac{\partial g}{\partial x_1}dx_1 + \cdots + \frac{\partial g}{\partial x_n}dx_n\Big) \\ &= df + kdg. \end{aligned}$$

k-forms and the Wedge Product

The definition of 2-forms, 3-forms, etc. relies on the *wedge* product for differential forms, denoted $\wedge$. Before we give the properties of $\wedge$, let's define k-forms ($k \geq 2$) in terms of $\wedge$.

Definition 24. Let $E \subseteq \mathbb{R}^n$ be open, and let $k \geq 2$. A ***k*-form** on E is an expression of the form

$$\sum_{i_1,\ldots,i_k=1}^{n} f_{i_1,\ldots,i_k}\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Example 78. For example,

$$xy\,dx \wedge dy + 4\,dy \wedge dx + e^{xy}\,dx \wedge dz - 5x\,dz \wedge dx - \sin^2(z)\,dy \wedge dz + \cos^2(z)\,dz \wedge dy + 5\,dy \wedge dy$$

is a 2-form on $\mathbb{R}^3$ (with coordinates x, y, z), and

$$\rho \sin(\theta)\,d\rho \wedge d\phi \wedge d\theta + e^{\rho} d\theta \wedge d\phi \wedge d\rho$$

is a 3-form on $\mathbb{R}^3$ (with coordinates ρ, ϕ, θ).

Remark 31. Just as for 1-forms, you should think of the space of k-forms as formed by all linear combinations of the **basic k-forms** $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ (where $i_1, \ldots, i_k$ each range from 1 to n). Sums of k-forms and multiplication of a k-form by a function is defined exactly as expected: if

$$\omega = \sum_{i_1,\ldots,i_k=1}^{n} f_{i_1,\ldots,i_k}\,dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \text{and} \quad \eta = \sum_{i_1,\ldots,i_k=1}^{n} g_{i_1,\ldots,i_k}\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

are k-forms and if $h : E \to \mathbb{R}$ is a function, then

$$\omega + \eta \stackrel{def}{=} \sum_{i_1,\ldots,i_k=1}^{n} (f_{i_1,\ldots,i_k} + g_{i_1,\ldots,i_k})\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and

$$h\omega \stackrel{def}{=} \sum_{i_1,\ldots,i_k=1}^{n} hf_{i_1,\ldots,i_k}\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The **wedge** product $\wedge$ allows one to multiply a k-form and an ℓ-form to get a $(k+\ell)$-form. In many ways it behaves exactly like one would expect from a product, but (unlike most products you are familiar with) it is *anticommutative*. In particular, if ω, η are k-forms on E and α, β are ℓ-forms on E, if ☺ is a m-form on E, and if $f : E \to \mathbb{R}$ is a function then

- (Distributivity) $(\omega + \eta) \wedge \alpha = \omega \wedge \alpha + \eta \wedge \alpha$ and $\omega \wedge (\alpha + \beta) = \omega \wedge \alpha + \omega \wedge \beta$.
- (Associativity) $(\omega \wedge \alpha) \wedge$ ☺ $= \omega \wedge (\alpha \wedge$ ☺$)$
- (Homogeneity) $(f\omega) \wedge \alpha = f(\omega \wedge \alpha) = \omega \wedge (f\alpha)$
- (Anticommutativity) $dx_i \wedge dx_j = -(dx_j \wedge dx_i)$. More generally, if $dx_{i_1} \wedge \cdots \wedge dxi_k$ is obtained from the basic k-form $dx_{j_1} \wedge \cdots \wedge dx_{j_k}$ by swapping a single pair of variables, then $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = -dx_{j_1} \wedge \cdots \wedge dx_{j_k}$.

These properties—especially anticommutativity—allow us to greatly simplify how we write differential forms.

Remark 32. Be careful with anticommutativity. For example, it is not necessarily the case that $\alpha \wedge \beta = -\beta \wedge \alpha$ for every k-form α and every ℓ-form β. For a counterexample, note that

$$(dx) \wedge (dy \wedge dz) = (dx \wedge dy) \wedge dz = -(dy \wedge dx) \wedge dz = -dy \wedge (dx \wedge dz) = dy \wedge (dz \wedge dx) = (dy \wedge dz) \wedge dx.$$

What is true (as you should show!) is that $\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$.

Remark 33. First, note that by anticommutativity, $dx_i \wedge dx_i = -(dx_i \wedge dx_i)$, so that $dx_i \wedge dx_i = 0 dx_i \wedge dx_i$. In particular, every basic k-form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ that contains a repeated variable is automatically the 0 k-form.

Moreover, by anticommutativity we can reorder a basic k-form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ so that $dx_{i_1}, \ldots, dx_{i_k}$ appear in whatever order we choose (at the cost, of course, of picking up a factor of -1 for every swap of perhaps, of introducing a factor of (-1) in the coefficient).

Example 79. For a concrete example of how this is done, we can simplify a previous example as follows:

$$\begin{aligned}
&xy\,dx \wedge dy + 4\underbrace{dy \wedge dx}_{-dx\wedge dy} + e^{xy}\underbrace{dx \wedge dz}_{-dz\wedge dx} - 5x\,dz \wedge dx - \sin^2(z)\,dy \wedge dz + \cos^2(z)\underbrace{dz \wedge dy}_{-dy\wedge dz} + \underbrace{5\,dy \wedge dy}_{0} \\
&= (xy - 4)\,dx \wedge dy + (-e^{xy} - 5x)\,dz \wedge dx + (-\sin^2(z) - \cos^2(z))\,dy \wedge dz \\
&= -1\,dy \wedge dz + (-e^{xy} - 5x)\,dz \wedge dx + (xy - 4)\,dx \wedge dy.
\end{aligned}$$

Indeed, *every* 2-form ω on $\mathbb{R}^3$ can be written in the form

$$\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$$

Similarly, every 3-form η on $\mathbb{R}^3$ can be written in the form

$$\eta = f\,dx \wedge dy \wedge dz,$$

and, more generally, every n-form γ on $\mathbb{R}^n$ can be written in the form

$$\gamma = G\,dx_1 \wedge \cdots \wedge dx_n.$$

Remark 34. Note that if $k > n$, then every basic k-form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ on $E \subseteq \mathbb{R}^n$ will have at least one repeated variable, and therefore can be arranged to contain a term of the form $dx_i \wedge dx_i$, and is therefore 0. For this reason, we only study k-forms on $\mathbb{R}^n$ for $0 \leq k \leq n$.

Lecture 19: More Differential Forms

Learning Objectives:

- Compute the differential of a function, and the exterior derivative of a differential form.
- Use the properties of the wedge product to manipulate differential forms.
- Related the exterior derivative to the gradient, curl, and divergence operations.
- Compute the pullback of a differential form.
- Relate integration of differential forms with the notions of integration we have seen earlier in the course.

We start today with a concrete example of how to use the wedge product to simplify a given differential form.

Example 80. For a concrete example of how this is done, we can simplify a previous example as follows:

$$\begin{aligned}
&xy\,dx\wedge dy + 4\underbrace{dy\wedge dx}_{-dx\wedge dy} + e^{xy}\underbrace{dx\wedge dz}_{-dz\wedge dx} - 5x\,dz\wedge dx - \sin^2(z)\,dy\wedge dz + \cos^2(z)\underbrace{dz\wedge dy}_{-dy\wedge dz} + \underbrace{5\,dy\wedge dy}_{0}\\
&= (xy-4)\,dx\wedge dy + (-e^{xy}-5x)\,dz\wedge dx + (-\sin^2(z)-\cos^2(z))\,dy\wedge dz\\
&= -1\,dy\wedge dz + (-e^{xy}-5x)\,dz\wedge dx + (xy-4)\,dx\wedge dy.
\end{aligned}$$

Indeed, *every* 2-form ω on $\mathbb{R}^3$ can be written uniquely in the form

$$\omega = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy.$$

A similar sort of "standard" representation exists for k-forms in open regions $E\subseteq\mathbb{R}^n$ for every n and $0\le k\le n$. For example, every 3-form η on $\mathbb{R}^3$ can be written uniquely in the form

$$\eta = f\,dx\wedge dy\wedge dz,$$

and, more generally, every n-form γ on $\mathbb{R}^n$ can be written uniquely in the form

$$\gamma = G\,dx_1\wedge\cdots\wedge dx_n,$$

after manipulating the differential form using properties of the wedge product.

The Exterior Derivative

The differential of a function can be extended to give a notion of differentiation on k-forms.

Definition 25. Let $E\subseteq\mathbb{R}^n$ be open. For a k-form $\omega=\sum_{i_1,\ldots,i_k=1}^n f_{i_1,\ldots,i_k}\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}$, we define the **exterior derivative** of ω, $d\omega$, to be the $(k+1)$-form given by

$$d\omega \overset{def}{=} \sum_{i_1,\ldots,i_k=1}^{n} df_{i_1,\ldots,i_k}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}.$$

Remark 35. The above definition is useful in practice, but it is worth noting that the operator d is characterized by the following two properties:

- $d(f\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and
- $d(\omega + \eta) = d\omega + d\eta$ for k-forms ω and η.

Example 81. To illustrate how to compute the exterior derivative, let's compute $d\omega$, where ω is the 3-form on $\mathbb{R}^4$ given by

$$\omega = xwz\,dx \wedge dy \wedge dw + 3\sin(xy)\,dy \wedge dz \wedge dw.$$

By applying the definition of the exterior derivative, we have

$$\begin{aligned} d\omega &= d(xwz\,dx \wedge dy \wedge dw + 3\sin(xy)\,dy \wedge dz \wedge dw) \\ &= d(xwz) \wedge dx \wedge dy \wedge dw + d(3\sin(xy)) \wedge dy \wedge dz \wedge dw \\ &= (yz\,dx + 0\,dy + xw\,dz + xz\,dw) \wedge dx \wedge dy \wedge dw \\ &\qquad + (3y\cos(xy)\,dx + 3x\cos(xy)\,dy + 0\,dz + 0\,dw) \wedge dy \wedge dz \wedge dw \\ &= yz\,\underbrace{dx \wedge dx \wedge dy \wedge dw}_{0} + xw\,dz \wedge dx \wedge dy \wedge dw + xz\,\underbrace{dw \wedge dx \wedge dy \wedge dw}_{0} \\ &\qquad 3y\cos(xy)\,dx \wedge dy \wedge dz \wedge dw + 3x\cos(xy)\,\underbrace{dy \wedge dy \wedge dz \wedge dw}_{0} \\ &= xw\,\underbrace{dz \wedge dx \wedge dy \wedge dw}_{=(-1)^2 dx\wedge dy\wedge dz\wedge dw} + 3y\cos(xy)\,dx \wedge dy \wedge dz \wedge dw \\ &= (wx + 3y\cos(xy))\,dx \wedge dy \wedge dz \wedge dw, \end{aligned}$$

where in the last step we used anticommutativity to compute that

$$dz \wedge dx \wedge dy \wedge dw = -dx \wedge dz \wedge dy \wedge dw = dx \wedge dy \wedge dz \wedge dw.$$

Grad, Div, and Curl: Reprise

In a previous lecture I mentioned that grad, div, and curl are special instances of "the derivative" of certain differential forms. To illustrate this, let's look at what happens in $\mathbb{R}^3$. We consider the following correspondences between differential forms and vector fields and functions.

$$\begin{aligned} &\text{0-Form}: \quad f && \sim \quad \text{Function}: \quad f \\ &\text{1-Form}: \quad Pdx + Qdy + Rdz && \sim \quad \text{Vector Field}: \quad P\vec{i} + Q\vec{j} + R\vec{k} \\ &\text{2-Form}: \quad Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy && \sim \quad \text{Vector Field}: \quad P\vec{i} + Q\vec{j} + R\vec{k} \\ &\text{3-Form}: \quad f\,dx \wedge dy \wedge dz && \sim \quad \text{Function}: \quad f \end{aligned}$$

With this correspondence, we see that

$$df = f_x dx + f_y dy + f_z dz \sim f_x\vec{i} + f_y\vec{j} + f_z\vec{k} = \nabla f,$$

so that the 1-form df corresponds to the vector field ∇f.

Similarly,

$$\begin{aligned}
&d(Pdx+Qdy+Rdz)\\
&\quad = dP\wedge dx + dQ\wedge dy + dR\wedge dz\\
&\quad = (P_x dx + P_y dy + P_z dz)\wedge dx\\
&\qquad\qquad + (Q_x dx + Q_y dy + Q_z dz)\wedge dy\\
&\qquad\qquad + (R_x dx + R_y dy + R_z dz)\wedge dz\\
&\quad = P_x dx\wedge dx + P_y dy\wedge dx + P_z dz\wedge dx\\
&\qquad\qquad + Q_x dx\wedge dy + Q_y dy\wedge dy + Q_z dz\wedge dy\\
&\qquad\qquad + R_x dx\wedge dz + R_y dy\wedge dz + R_z dz\wedge dz\\
&\quad = -P_y dx\wedge dy + P_z dz\wedge dx\\
&\qquad\qquad + Q_x dx\wedge dy - Q_z dy\wedge dz\\
&\qquad\qquad - R_x dz\wedge dx + R_y dy\wedge dz\\
&\quad = (R_y - Q_z)dy\wedge dz + (P_z - R_x)dz\wedge dx + (Q_x - P_y)dx\wedge dy\\
&\quad \sim (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}\\
&\quad = \operatorname{curl}(P\vec{i}+Q\vec{j}+R\vec{k}),
\end{aligned}$$

so that the 2-form $d(Pdx+Qdy+Rdz)$ corresponds to the vector field $\operatorname{curl}(P\vec{i}+Q\vec{j}+R\vec{k})$.

Finally, we have (shortening the computation by ignoring the terms that become 0 by antisymmetry)

$$\begin{aligned}
&d(Pdy\wedge dz + Qdz\wedge dx + Rdx\wedge dy)\\
&\qquad = dP\wedge dy\wedge dz + dQ\wedge dz\wedge dx + dR\wedge dx\wedge dy\\
&\qquad = P_x dx\wedge dy\wedge dz + Q_y dy\wedge dz\wedge dx + R_z dz\wedge dx\wedge dy\\
&\qquad = P_x dx\wedge dy\wedge dz - Q_y dy\wedge dx\wedge dz - R_z dx\wedge dz\wedge dy\\
&\qquad = P_x dx\wedge dy\wedge dz + Q_y dx\wedge dy\wedge dz + R_z dx\wedge dy\wedge dz\\
&\qquad = \Big(P_x + Q_y + R_z\Big)dx\wedge dy\wedge dz\\
&\qquad \sim P_x + Q_y + R_z\\
&\qquad = \operatorname{div}(P\vec{i}+Q\vec{j}+R\vec{k}),
\end{aligned}$$

so that the 3-form $d(Pdy\wedge dz + Qdz\wedge dx + Rdx\wedge dy)$ corresponds to the function $\operatorname{div}(P\vec{i}+Q\vec{j}+R\vec{k})$.

Remark 36. We have shown that if $\vec{F}:\mathbb{R}^3\to\mathbb{R}^3$ is a C^2 vector field on $\mathbb{R}^3$ and if $f:\mathbb{R}^3\to\mathbb{R}$ is a C^2 function on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl}(\vec{F}))=0$ and $\operatorname{curl}(\operatorname{grad}(f))=\vec{0}$. In light of the above discussion, and setting η to be the 1-form on $\mathbb{R}^3$ corresponding to $\vec{F}$, we have

$$d(d\eta)=0 \quad\text{and}\quad d(df)=0$$

as differential forms. The computations here boiled down to applications of Clairaut's Theorem. The somewhat surprising fact is that this is true in general: if ω is a C^2 k-form on $\mathbb{R}^n$, then $d(d\omega)=0$. This fact, often stated with the shorthand that $d^2=0$, is proved on your homework.

Pullbacks of Differential Forms

Suppose that $D\subseteq\mathbb{R}^n$ (with coordinates $(u_1,\ldots,u_n)$) and $E\subseteq\mathbb{R}^m$ (with coordinates $(x_1,\ldots,x_m)$) are open, and that

$$T:D\to E,\ T(u_1,\ldots,u_n)=(x_1(u_1,\ldots,u_n),\ldots,x_m(u_1,\ldots,u_n))$$

is a C^1 map. Then recall that for a function $f : E \to \mathbb{R}$, we called $f \circ T : D \to \mathbb{R}$ the **pullback** of the function f under T. This operation "pulled back" the domain of f from E to D by precomposing f with $T : D \to E$.

This definition can be extended to define the pullback of k-forms on E under T. Because we will think of "pulling back" a differential form as an operation, we will introduce some fresh notation to make computing these pullbacks slightly easier.

Definition 26. Suppose that $D \subseteq \mathbb{R}^n$ (with coordinates $(u_1, \ldots, u_n)$) and $E \subseteq \mathbb{R}^m$ (with coordinates $(x_1, \ldots, x_m)$) are open, and that

$$T : D \to E,\ T(u_1, \ldots, u_n) = (x_1(u_1, \ldots, u_n), \ldots, x_m(u_1, \ldots, u_n))$$

is a C^1 map.

- Let $f : E \to \mathbb{R}$ be a 0-form on E. The **pullback** of f by T, T^*f, is the 0-form on D defined by $T^*f(\vec{u}) \stackrel{def}{=} f(T(\vec{u}))$.
- Fix $j = 1, \ldots, m$. The **pullback** of the basic 1-form dx_j on E, $T^*(dx_j)$, is the 1-form on D defined by
$$T^*(dx_j) \stackrel{def}{=} d(x_j(u_1, \ldots, u_n)) = \frac{\partial x_j}{\partial u_1} du_1 + \cdots + \frac{\partial x_j}{\partial u_n} du_n.$$
- Suppose that $\omega = \sum f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is a k-form on E. The **pullback** of ω by T, $T^*\omega$, is the k-form on D defined by
$$T^*\omega = T^*\left(\sum_{i_1,\ldots,i_k=1}^{n} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \stackrel{def}{=} \sum_{i_1,\ldots,i_k=1}^{n} T^* f_{i_1,\ldots,i_k} (T^* dx_{i_1}) \wedge \cdots \wedge (T^* dx_{i_k}).$$

The previous definition can be a little intimidating, but in practice it is very "computationally efficient". Here is an example.

Example 82. Compute $P^*(dx \wedge dy)$, where $P : \mathbb{R}^2 \to \mathbb{R}^2$ is the map $P(r, \theta) = (r\cos(\theta), r\sin(\theta))$.

Note that we are essentially asked to write the 2-form $dx \wedge dy$ on $\mathbb{R}^2$ in terms of polar coordinates (r, θ). We should end up with a 2-form in terms of r, θ, and dr and $d\theta$. To this end, we have (and omitting the terms that will be 0 because of antisymmetry)

$$\begin{aligned} P^*(dx \wedge dy) &= d(x(r,\theta)) \wedge d(y(r,\theta)) \\ &= (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \\ &= r\cos^2(\theta)dr \wedge d\theta - r\sin^2(\theta)d\theta \wedge dr \\ &= (r\cos^2(\theta) + r\sin^2(\theta))dr \wedge d\theta \\ &= r\, dr \wedge d\theta. \end{aligned}$$

This looks suspiciously like the $P^*(dA(x,y)) = r\, dA(r,\theta)$ relationship that we observed when discussing change of variables. Indeed, if we didn't simplify the partial derivatives of x and y with respect to r and θ in the computation above, we would have that

$$P^*(dx \wedge dy) = (x_r y_\theta - x_\theta y_r)dr \wedge d\theta = (\det(DT(r,\theta)))dr \wedge d\theta.$$

The relationship between the pullback of a differential form and determinants seen in the last example generalizes to higher dimensions. Indeed, you will show on your homework that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 with

$$T(u_1, \ldots, u_n) = (x_1(u_1, \ldots, u_n), \ldots, x_n(u_1, \ldots, u_n)),$$

then

$$T^*(dx_1 \wedge \cdots \wedge dx_n) = (\det(DT(u_1, \ldots, u_n)))du_1 \wedge \cdots \wedge du_n.$$

Example 83. To illustrate this claim, let's compute $T^*(dx \wedge dy \wedge dz)$, where $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the C^1 spherical coordinate map given by

$$T(\rho, \phi, \theta) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)).$$

According to the result from your homework, we expect that

$$T^*(dx \wedge dy \wedge dz) = (\det(DT(\rho, \phi, \theta)))d\rho \wedge d\phi \wedge d\theta = \rho^2 \sin(\phi)\, d\rho \wedge d\phi \wedge d\theta.$$

To verify this, we compute (again ignoring the terms that will be 0 by antisymmetry) as

$$\begin{aligned}
T^*(dx \wedge dy \wedge dz) &= d(\rho\cos(\theta)\sin(\phi)) \wedge d(\rho\sin(\theta)\sin(\phi)) \wedge d(\rho\cos(\phi)) \\
&= (\cos(\theta)\sin(\phi)d\rho + \rho\cos(\theta)\cos(\phi)d\phi - \rho\sin(\theta)\sin(\phi)d\theta) \\
&\quad\ \wedge (\sin(\theta)\sin(\phi)d\rho + \rho\sin(\theta)\cos(\phi)d\phi + \rho\cos(\theta)\sin(\phi)d\theta) \\
&\quad\ \wedge (\cos(\phi)d\rho - \rho\sin(\phi)d\phi + 0d\theta) \\
&= -\rho^2\cos^2(\theta)\sin^3(\phi)d\rho \wedge d\theta \wedge d\phi + \rho^2\cos^2(\theta)\cos^2(\phi)\sin(\phi)d\phi \wedge d\theta \wedge d\rho \\
&\quad\ + \rho^2\sin^2(\theta)\sin^3(\phi)d\theta \wedge d\rho \wedge d\phi - \rho^2\sin^2(\theta)\cos^2(\phi)\sin(\phi)d\theta \wedge d\phi \wedge d\rho \\
&= \rho^2\cos^2(\theta)\sin^3(\phi)d\rho \wedge d\phi \wedge d\theta + \rho^2\cos^2(\theta)\cos^2(\phi)\sin(\phi)d\rho \wedge d\phi \wedge d\theta \\
&\quad\ + \rho^2\sin^2(\theta)\sin^3(\phi)d\rho \wedge d\phi \wedge d\theta + \rho^2\sin^2(\theta)\cos^2(\phi)\sin(\phi)d\rho \wedge d\phi \wedge d\theta \\
&= \Big(\rho^2\cos^2(\theta)\sin^3(\phi) + \rho^2\cos^2(\theta)\cos^2(\phi)\sin(\phi) \\
&\qquad\quad + \rho^2\sin^2(\theta)\sin^3(\phi) + \rho^2\sin^2(\theta)\cos^2(\phi)\sin(\phi)\Big)d\rho \wedge d\phi \wedge d\theta \\
&= \Big(\rho^2\sin^3(\phi) + \rho^2\cos^2(\phi)\sin(\phi)\Big)d\rho \wedge d\phi \wedge d\theta \\
&= \rho^2\sin(\phi)\, d\rho \wedge d\phi \wedge d\theta,
\end{aligned}$$

exactly as expected!

In the previous examples, we were concerned with pulling back n-forms using a map $T : \mathbb{R}^n \to \mathbb{R}^n$. More generally, we will also be interested in pulling back k-forms from $\mathbb{R}^n$ to $\mathbb{R}^k$ via a map $T : \mathbb{R}^k \to \mathbb{R}^n$. We will see specific instances of this when we discuss vector line integrals (i.e. integrals of 1-forms over oriented curves) and vector surface integrals (i.e. integrals of 2-forms over oriented surfaces).

Lecture 20: Integration of Differential Forms and Line Integrals

Learning Objectives:

- Relate integration of differential forms with the notions of integration we have seen earlier in the course.
- Determine whether a parametrization of a smooth oriented curve preserves or reverses orientation.

Integration of Differential Forms

As I mentioned, differential forms are "things that we can integrate". To this end, we have the following definition.

Definition 27. Suppose that $\omega = f(\vec{x})dx_1 \wedge \cdots \wedge dx_n$ is a continuous n-form on (an open set containing) a bounded region $D \subset \mathbb{R}^n$ such that ∂D has measure zero. Then we define

$$\int_D \omega = \int_D f(\vec{x})dx_1 \wedge \cdots \wedge dx_n \stackrel{def}{=} \int_D f(\vec{x})\, dV_n(\vec{x}).$$

The order of the variables is important here in order to avoid any ambiguity about what the value of the integral is.

Example 84. Let's investigate how the Change of Variables Theorem interacts with differential forms. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1, injective map with $DT(\vec{u})$ invertible that sends an elementary region $D \subset \mathbb{R}^n$ onto a region $T(D) \subset \mathbb{R}^n$, and if $\omega = f(\vec{x})dx_1 \wedge \cdots \wedge dx_n$ is a n-form on $T(D)$, then we have

$$\begin{aligned}\int_{T(D)} \omega &= \int_{T(D)} f(\vec{x})\, dV_n(\vec{x}) \\ &= \int_D f(T(\vec{u}))|\det(DT(\vec{u}))|\, dV_n(\vec{u}) \\ &= \int_D f(T(\vec{u}))|\det(DT(\vec{u}))|\, du_1 \wedge \cdots \wedge du_n.\end{aligned}$$

The right-hand-side is not exactly the hoped-for

$$\int_D T^*\omega = \int_D f(T(\vec{u}))(\det(DT(\vec{u})))du_1 \wedge \cdots \wedge du_n.$$

However, as long as $\det(DT(\vec{u}))$ is either always positive or always negative, then we can express the change of variables theorem purely in terms of differential forms. To do this, we make a definition.

Definition 28. Suppose $D \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ are open, and that $T : D \to E$ is a C^1, bijective map. We say that T is **orientation preserving** if $\det(DT(\vec{u})) > 0$ for every $\vec{u} \in D$, and **orientation reversing** if $\det(DT(\vec{u})) < 0$ for every $\vec{u} \in D$.

In light of this definition, we have that

$$\int_{T(D)} \omega = \int_D T^*\omega \quad \text{if } T \text{ is orientation preserving,}$$

and

$$\int_{T(D)} \omega = -\int_D T^*\omega \quad \text{if } T \text{ is orientation reversing.}$$

Example 85. This phenomenon is reflected in the single-variable calculus substitution observation where the bounds on the integral get "flipped" when you make the subsitution. That is, if $x : [a,b] \to [c,d]$ is a C^1 bijective map with $x'(u) < 0$ throughout $[a,b]$, then $[c,d] = x([a,b])$ with $x(a) = d$ and $x(b) = c$, so that

$$\int_{T([a,b])} f(x)\,dx = \int_c^d f(x)\,dx = \int_b^a \underbrace{f(x(u))x'(u)\,du}_{x^*(f(x)\,dx)} = -\int_a^b x^*(f(x)dx) = -\int_{[a,b]} x^*(f(x)\,dx).$$

Because $\det(Dx(u)) = x'(u) < 0$ throughout $[a,b]$, the map $x(u)$ is orientation reversing.

Over the next few days will will discuss how to integrate differential forms over certain subsets of $\mathbb{R}^n$ (like curves and surfaces). In particular, we will integrate 1-forms over curves, and 2-forms over surfaces. In each case, we now need to consider smooth curves and surfaces to be oriented. For smooth curves, this means that we will need to establish which direction on the curve is "forwards" (this is done by continuously specifying a unit tangent vector $\vec{T}$ at each point on the curve). For smooth surfaces, we will need to establish which direction is "up" from the surface (this is done by continuously specifying unit normal vectors to the surface that point in the "up" direction). More details to come!

To What End?

For the rest of the course, we will attempt to study several generalizations of the Fundamental Theorem of Calculus. Each of these theorems can be expressed in the following form:

$$\int_E d\omega = \int_{\partial E} \omega$$

This is an elegant formula that requires a lot of unpacking. Here $E \subset \mathbb{R}^n$ and ∂E represents the *geometric boundary* of E. This is not necessarily the same as the topological boundary that we discussed last quarter. For example, if $E \subset \mathbb{R}^3$ is a surface, then ∂E is the curve (or union of curves) that form the "edge" of the surface.

Example 86. As a concrete instance of this framework, consider the Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

To view this in the framework above, we think of $E = [a,b]$ as an interval in $\mathbb{R}$ oriented "forwards", so that the initial point of E is a and the final point of E is b. Here $f'(x)dx = df$, where f is a C^1 0-form on E. Finally, $f(b) - f(a) = (+1)f(b) + (-1)f(a)$ is the "integral" of the 0-form f over set $\partial E = \{a,b\}$ (where the $+1$ indicates that b is the final point of E, and the -1 indicates that a is the initial point of E). With this framework,

$$\int_a^b f'(x)\,dx = f(b) - f(a) \quad \Rightarrow \quad \int_E df = \int_{\partial E} f.$$

Each time we discuss one of the generalizations of the Fundamental Theorem of Calculus, we will discuss how it fits into this framework.

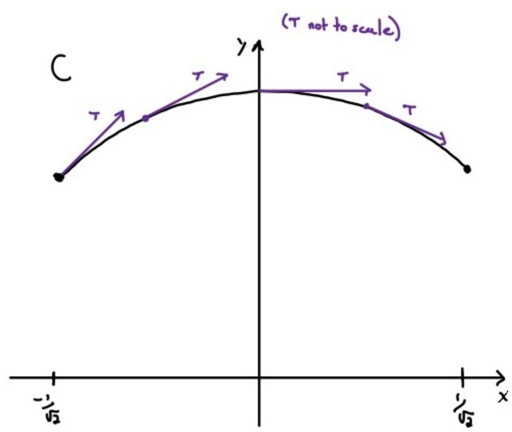
Orientation of Curves

We now develop a notion of line integral in terms of the pullback of a differential form. Once we do this, we will be able to establish an interpretation of these line integrals in terms of vector fields. Because of this, these will be called *vector line integrals* to distinguish them from *scalar line integrals*. We will clearly link vector line integrals to scalar line integrals, and it is there that we will see how the orientation of the curve we are integrating over plays a role in the evaluation of the integral. Before we dive into this, let's remind ourselves of the definition of oriented smooth curve, and explore how orientations and parametrizations are related.

Definition 29. Let $C \subset \mathbb{R}^n$ be a smooth curve that does not intersect itself (except possibly at its endpoints)[18]. An **orientation** of C is a continuous choice of unit tangent vector $\vec{T}$ on C.

An orientation of C is intended to specify in what direction C is traced out, in the sense that if C is determined by the motion of a particle then the unit tangent vector $\vec{T}$ at each point on C points in the direction of the particle's motion at that instant.

Example 87. Consider the portion C of the unit circle $x^2 + y^2 = 1$ with $y \geq \frac{1}{\sqrt{2}}$, and give C the "left-to-right" orientation.



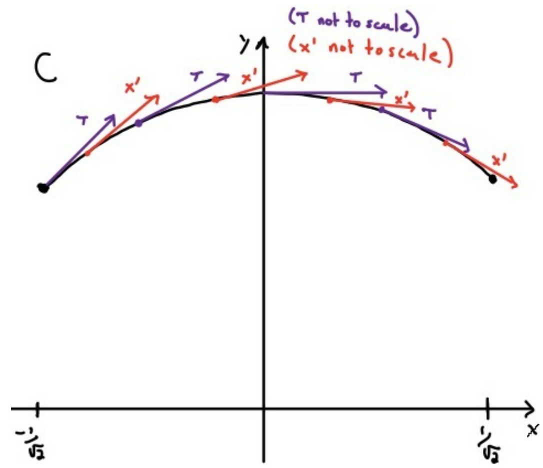
Then the parametrization $\vec{x}(t) = (t, \sqrt{1-t^2})$, $-\frac{1}{\sqrt{2}} \leq t \leq \frac{1}{\sqrt{2}}$ parametrizes C, and

$$\vec{x}\,'(t) = \begin{bmatrix} 1 \\ -\frac{t}{\sqrt{1-t^2}} \end{bmatrix}$$

has positive x-component, and therefore points "rightward" at each point. Therefore the direction in which $\vec{x}$ traces out C agrees with the orientation of C, and the unit tangent vector to C at each point $\vec{x}(t)$ is given by

$$\vec{T}(t) = \frac{1}{\|\vec{x}\,'(t)\|}\vec{x}\,'(t) = \frac{1}{\sqrt{1+\frac{t^2}{1-t^2}}}\begin{bmatrix} 1 \\ -\frac{t}{\sqrt{1-t^2}} \end{bmatrix} = \sqrt{1-t^2}\begin{bmatrix} 1 \\ -\frac{t}{\sqrt{1-t^2}} \end{bmatrix} = \begin{bmatrix} \sqrt{1-t^2} \\ -t \end{bmatrix}.$$

[18] A curve that doesn't intersect itself (except possible at its endpoints) is called **simple**. If a curve can be parametrized by a path that starts and ends at the same point, then it is called **closed**.



Because the tangent vectors induced by $\vec{x}$ agree with the orientation of C, we call the parametrization $\vec{x}$ of C **orientation preserving**.

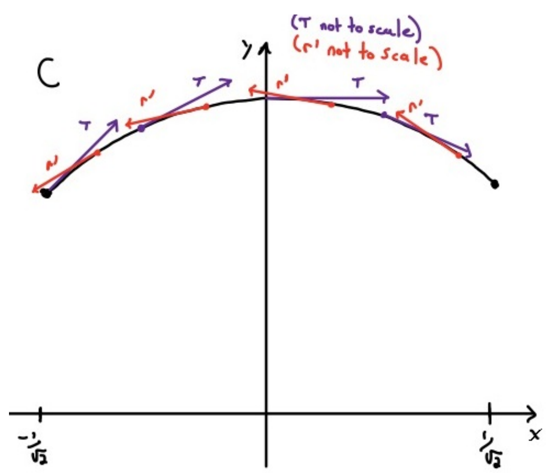
On the other hand, consider the parametrization $\vec{r}(\theta) = (\cos(\theta), \sin(\theta)),\ \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ of C. Note that

$$\vec{r}\,'(\theta) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

has negative x-component for each $\theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, the tangent vectors induced by $\vec{r}$ point "leftwards" at each point, and therefore the unit tangent vector

$$\frac{1}{\|\vec{r}\,'(\theta)\|}\vec{r}\,'(\theta) = \frac{1}{1}\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

is exactly the *negative* of the unit tangent vector specified by the orientation. Therefore we say that the parametrization $\vec{r}$ of C is **orientation reversing**.



In particular examples, one should carefully check to ensure that a parametrization of an oriented curve is orientation preserving.

Lecture 21: More Line Integrals

Learning Objectives:

- Develop the notion of vector line integral via the pullback of a differential 1-form.
- Explore the basic properties and interpretations of vector line integrals.

Just as scalar line integrals were defined in terms of a "change of variables" type formula (which we now recognize as a sort of pullback), integration of 1-forms and (their interpretation as) vector line integrals are also defined in terms of pullbacks. We start by defining the integral of a 1-form.

Definition 30. Let $E \subseteq \mathbb{R}^n$ be an open set and ω a continuous 1-form on E, and suppose that C is a smooth oriented curve in E. Let $\vec{x} : [a, b] \to C$ be an orientation-preserving C^1 parametrization of C with $\vec{x}\,'(t) \neq \vec{0}$ for each $t \in [a, b]$. Then we define the **integral of ω over C** to be

$$\int_C \omega \stackrel{def}{=} \int_{[a,b]} \vec{x}^* \omega.$$

For example, if $n = 3$, $\omega = Pdx + Qdy + Rdz$, and $\vec{x}(t) = (x(t), y(t), z(t))$, then

$$\begin{aligned}\int_C \omega &= \int_C Pdx + Qdy + Rdz \\ &= \int_a^b \Big(P(\vec{x}(t))x'(t)dt + Q(\vec{x}(t))y'(t)dt + R(\vec{x}(t))z'(t)dt\Big) \\ &= \int_a^b (P(\vec{x}(t))x'(t) + Q(\vec{x}(t))y'(t) + R(\vec{x}(t))z'(t))dt.\end{aligned}$$

The result is a single-variable integral, and can therefore be approached using techniques from single-variable calculus.

To link this approach with vector fields, consider that integration of $Pdx+Qdy+Rdz$ over C should somehow correspond to a type of "vector line integral" of $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ over C. But the integrand of the pullback is exactly $\vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)$. Indeed, if we take this reasoning a few steps further we can have

$$\begin{aligned}
\int_C \omega &= \int_a^b (P(\vec{x}(t))x'(t) + Q(\vec{x}(t))y'(t) + R(\vec{x}(t))z'(t))dt \\
&= \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt \\
&= \int_a^b \vec{F}(\vec{x}(t)) \cdot \underbrace{\left(\frac{1}{\|\vec{x}\,'(t)\|}\vec{x}\,'(t)\right)}_{=\vec{T}(\vec{x}(t))} \|\vec{x}\,'(t)\|\, dt \\
&= \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{T}(\vec{x}(t)) \|\vec{x}\,'(t)\|\, dt \\
&= \int_C \vec{F} \cdot \vec{T}\, ds,
\end{aligned}$$

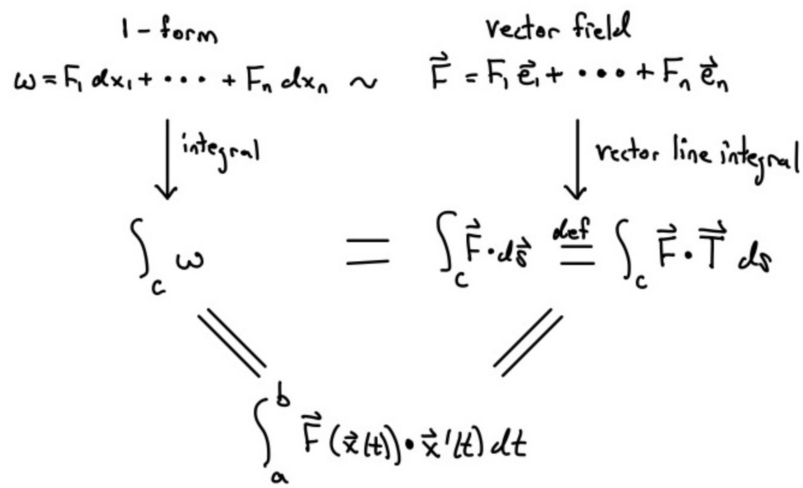
so that the integral of the 1-form $\omega = Pdx + Qdy + Rdz$ over C is equal to the scalar line integral of the function $\vec{F} \cdot \vec{T}$ over C, where $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ and $\vec{T}$ is the unit tangent vector at each point of C that is specified by the orientation.

This same reasoning can be extended to curves (and vector fields) in $\mathbb{R}^n$, $n = 2, 3, 4, \ldots$, which leads us to make the following definition.

Definition 31. Let $E \subseteq \mathbb{R}^n$ be open and $\vec{F} : E \to \mathbb{R}^n$ a continuous vector field on E, and suppose that C is a smooth oriented curve in E. We define the **vector line integral** of $\vec{F}$ over C to be

$$\int_C \vec{F} \cdot d\vec{s} \overset{def}{=} \int_C \vec{F} \cdot \vec{T}\, ds.$$

Remark 37. To summarize, for an oriented smooth curve C in $\mathbb{R}^n$ and a smooth orientation-preserving parametrization $\vec{x} : [a, b] \to C$ of C, we have the following correspondence:



Example 88. Let C be the unit circle $x^2 + y^2 = 1$, oriented counterclockwise. Compute

$$\int_C \vec{F} \cdot d\vec{s},$$

where $\vec{F} = \frac{-y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}$.

To accomplish this, we parametrize C with $\vec{x}(t) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$. Note that since

$$\vec{x}'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

is the counterclockwise rotation of $\vec{x}(t) = (\cos(t), \sin(t))$ by $\frac{\pi}{2}$ radians, $\vec{x}\,(t)'$ does indeed point in the counterclockwise direction, and therefore $\vec{x}$ is orientation-preserving. Moreover, since $\|\vec{x}\,'(t)\| = 1$ at each t, we have $\vec{x}\,'(t) = \vec{T}(t)$. Therefore we have

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{s} &= \int_0^{2\pi} \vec{F}(\cos(t), \sin(t)) \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} dt \\
&= \int_0^{2\pi} \Big(\frac{-\sin(t)}{\sin^2(t) + \cos^2(t)} (-\sin(t)) + \frac{\cos(t)}{\sin^2(t) + \cos^2(t)} \cos(t) \Big)\, dt \\
&= \int_0^{2\pi} 1\, dt \\
&= 2\pi.
\end{aligned}$$

Remark 38. If the smooth curve C (with its orientation) are specified by a parametrization $\vec{x}$, then it is sometimes customary to simply write (for brevity)

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} \overset{notation}{=} \int_C \vec{F} \cdot d\vec{s}.$$

Remark 39. Note that if $\vec{r} : [c, d] \to C$ is an orientation-*reversing* parametrization and if $\vec{x} : [a, b] \to C$ is an orientation-preserving orientation, then if $u \in [c, d]$ and $t \in [a, b]$ satisfy $\vec{r}(u) = \vec{x}(t)$, then the unit tangent vectors to C arising from $\vec{r}$ and $\vec{x}$ at this point satisfy

$$\frac{1}{\|\vec{r}\,'(u)\|}\vec{r}\,'(u) = -\frac{1}{\|\vec{x}\,'(t)\|}\vec{x}\,'(t) = -\vec{T}(\vec{x}(t)) = -\vec{T}(\vec{r}(u)),$$

so that

$$\int_{\vec{r}} \vec{F} \cdot d\vec{s} = \int_c^d \vec{F}(\vec{r}(u)) \cdot \vec{r}\,'(u)\, du = -\int_c^d \vec{F}(\vec{r}(u)) \cdot \vec{T}(\vec{r}(u)) \|\vec{r}\,'(u)\|\, du = -\int_C \vec{F} \cdot d\vec{s}.$$

Therefore, if you accidentally parametrize an oriented curve in the incorrect direction while computing a vector line integral, then the value of the integral will be off by a factor of -1!

Remark 40. Note that since scalar line integrals are well-defined (i.e. they do not depend on the parametrization we choose), the vector line integral

$$\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot \vec{T}\, ds$$

is also well-defined. If ω is the 1-form corresponding to $\vec{F}$, then because we have

$$\int_C \omega = \int_C \vec{F} \cdot d\vec{s}$$

whenever we use an orientation-preserving parametrization of C, it is also the case that integrals of 1-forms do not depend on which orientation-preserving parametrization of the oriented curve we use.

Physical Interpretations

Remark 41. The

$$\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt$$

allows us a physical interpretation of the meaning of the vector line integral of the vector field $\vec{F}$ over the oriented curve $\vec{C}$, where $\vec{x} : [a, b] \to C$ is an orientation-preserving parametrization. Because we can think of $\vec{F}$ as a force field, and interpret $\vec{x}\,'(t)$ as the velocity of a particle tracing out C in the direction specified by the orientation, then we can interpret $\vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)$ as a measurement of the work being done on the particle by $\vec{F}$ when the particle is located at $\vec{x}(t)$. Therefore, the integral above can be interpreted as the total amount of **work** done by $\vec{F}$ on a particle traversing C.

Remark 42. When $\vec{F}$ is interpreted as the velocity field of a fluid, then one can also interpret $\vec{F} \cdot \vec{T}$ as a measurement of how much of the fluid flow at a point on C is "in the direction of $\vec{T}$". Therefore,

$$\int_C \vec{F} \cdot d\vec{s} = \int_C \vec{F} \cdot \vec{T}\, ds$$

can be interpreted as a net measurement of how much fluid flow over C is "in the direction of the orientation of C". If this quantity is positive, then (on balance) more fluid flow is happening "with" rather than "against" the motion of a particle traversing C. If the quantity is negative, then the opposite interpretation holds.

When C is a closed curve, then the integral above is called the **circulation of** $\vec{F}$ along the oriented closed curve C.

Remark 43. To see this interpretation of vector line integral as circulation in action, consider the example where C is the counterclockwise-oriented unit circle and $\vec{F} = \frac{-y}{x^2+y^2}\,\vec{\imath} + \frac{x}{x^2+y^2}\,\vec{\jmath}$. Note that the vector field $\vec{F}$ describes the motion of a fluid that is rotating (counterclockwise) around the origin along circles centered at the origin. Therefore our computation that

$$\int_C \vec{F} \cdot d\vec{s} = 2\pi$$

agrees with our intuition that the fluid describe by $\vec{F}$ is (on balance) flowing "with" the oriented curve C rather than "against" the orientation of C.

If $-C$ denotes[19] the unit circle with the *clockwise* orientation, then we would have

$$\int_{-C} \vec{F} \cdot d\vec{s} = -2\pi < 0,$$

which would indicate that now the fluid flows (on balance) "against" the orientation of $-C$ rather that "with" the orientation of C.

[19]Given an oriented curve C, it is common to write $-C$ for the curve C with reversed orientation.

Lecture 22: The Fundamental Theorem of Line Integrals

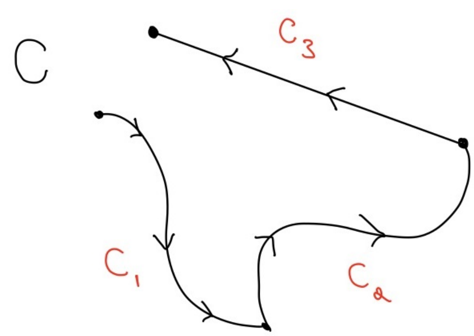
Learning Objectives:

- Integrate vector fields and functions over piecewise-smooth oriented curves.
- Use the Fundamental Theorem of Line Integrals to evaluate the line integrals of conservative fields.

Piecewise-Smooth Curves

Remark 44. Going forward, we will need to integrate vector fields over curves that arise as the geometric boundaries of regions in $\mathbb{R}^2$ or surfaces in $\mathbb{R}^3$. It will be too restrictive to limit ourselves to smooth curves, but with almost no additional effort we can extend our definitions to handle the case where a curve is *piecewise smooth*, which simply means that it consists of a finite union of smooth curves laid end-to-end. Here is a definition.

Definition 32. Say a piecewise-smooth curve $C \subset \mathbb{R}^n$ is **oriented** if there are smooth oriented curves $C_1, \ldots, C_k$ such that $C = C_1 \cup \cdots \cup C_k$, and where the ending point of C_i is the starting point of C_{i+1} for each $1 \leq i \leq k-1$. The **orientation** on C is taken on each C_i to be the orientation on C_i (but might be undefined at each of the endpoints of C_i), and we call the starting point of C_1 and the ending point of C_k the starting and ending points of C, respectively.



Remark 45. Note that if $C = C_1 \cup \cdots \cup C_k$ is a oriented piecewise-smooth curve, then we can reverse the orientation on C by reversing the orientation on each of $C_1, \ldots, C_k$. Using $-$ to denote a curve with its orientation reversed, this would mean that $-C = (-C_k) \cup \cdots \cup (-C_1)$.

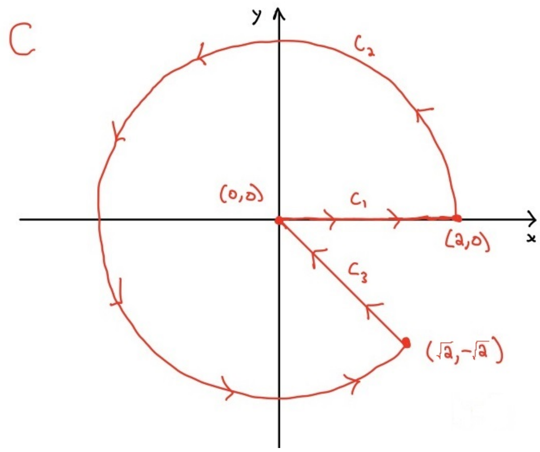
We define integration (including vector line integrals, scalar line integrals, and arc length) on piecewise smooth curves in terms of a sum of integrals over each smooth portion of the curve.

Remark 46. If $C \subset \mathbb{R}^n$ is an oriented piecewise-smooth curve with $C = C_1 \cup \cdots \cup C_k$, and if $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on an open set containing C, then we define the **vector line integral** of $\vec{F}$ over C to be

$$\int_C \vec{F} \cdot d\vec{s} \overset{def}{=} \int_{C_1} \vec{F} \cdot d\vec{s} + \cdots + \int_{C_k} \vec{F} \cdot d\vec{s}.$$

We define integrals of 1-forms over C and scalar line integrals over C similarly.

Example 89. Let $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $\vec{F} = 2xy^2e^{x^2y^2}\vec{i} + (2x^2ye^{x^2y^2} + \cos(y))\vec{j}$, and let C be the oriented piecewise-smooth curve consisting of the line segment starting at $(0,0)$ and ending at $(2,0)$, followed by the portion of the circle $x^2+y^2=4$ connecting $(2,0)$ to $(\sqrt{2},-\sqrt{2})$ (oriented counterclockwise, so this is 7/8 of a circle), followed by the line segment starting at $(\sqrt{2},-\sqrt{2})$ and ending at $(0,0)$.



We write $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from $(0,0)$ to $(2,0)$, C_2 is the portion of the circle, and C_3 is the line segment from $(\sqrt{2},-\sqrt{2})$ to $(0,0)$. We can parametrize each of these smooth curves with

$$\vec{r}_1(t) = (t,0),\ 0 \le t \le 2,$$
$$\vec{r}_2(t) = (2\cos(t), 2\sin(t)),\ 0 \le t \le \frac{7\pi}{4},$$
$$\vec{r}_3(t) = (\sqrt{2}-t, t-\sqrt{2}),\ 0 \le t \le \sqrt{2}.$$

We can therefore write

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{s} &= \int_{C_1} \vec{F}\cdot d\vec{s} + \int_{C_2} \vec{F}\cdot d\vec{s} + \int_{C_3} \vec{F}\cdot d\vec{s} \\ &= \int_0^2 \vec{F}(\vec{r}_1(t))\cdot \vec{r}_1{}'(t)\,dt + \int_0^{7\pi/4} \vec{F}(\vec{r}_2(t))\cdot \vec{r}_2{}'(t)\,dt + \int_0^{\sqrt{2}} \vec{F}(\vec{r}_3(t))\cdot \vec{r}_3{}'(t)\,dt \\ &= (\text{an unholy mess}).\end{aligned}$$

Rather than evaluate (an unholy mess) right now, let's try to find a more elegant way of computing the integral than brute-force computation. First, note that since[20]

$$\operatorname{curl}\vec{F} = (2x^2ye^{x^2y^2} + \cos(y))_x - (2xy^2e^{x^2y^2})_y = 4xye^{x^2y^2} + 4x^3y^3e^{x^2y^2} - 4xye^{x^2y^2} - 4x^3y^3e^{x^2y^2} = 0$$

throughout (the simply connected set) $\mathbb{R}^2$, $\vec{F}$ is conservative on $\mathbb{R}^2$ by Poincaré's Lemma. That is, there is a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla f = \vec{F}$.

To find f, note that we must have $f_x(x,y) = 2xy^2e^{x^2y^2}$ so that, taking an antiderivative in x, we must have $f(x,y) = e^{x^2y^2} + C(y)$ where $C(y)$ is some function that does not depend on x, but might depend on y. But then we should have

$$2x^2ye^{x^2y^2} + \cos(y) = f_y(x,y) = 2x^2ye^{x^2y^2} + C'(y),$$

[20]Here we use the scalar curl, since $\vec{F}$ is a 2-dimensional vector field.

so that $C'(y) = \cos(y)$, and therefore $C(y) = \sin(y) + k$ for some constant k. In other words, the potential function f must have the form $f(x, y) = e^{x^2y^2} + \sin(y) + k$ for a constant k. Taking $k = 0$ for simplicity, we see that

$$\nabla f = \nabla(e^{x^2y^2} + \sin(y)) = \vec{F}$$

as desired.

Because $\nabla f = \vec{F}$, the potential function f is a sort of antiderivative of $\vec{F}$. In single-variable calculus, the Fundamental Theorem of Calculus allows us to compute the integrals of continuous functions in terms of an antiderivative of the function. We might hope that something analogous holds for line integrals, and in searching for such a result we arrive at our first generalization of the Fundamental Theorem of Calculus.

Theorem 13 (The Fundamental Theorem of Line Integrals)**.** Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be C^1, and let $C \subset U$ be an oriented piecewise-smooth curve with starting point $\vec{a}$ and ending point $\vec{b}$. Then

$$\int_C \nabla f \cdot d\vec{s} = f(\vec{b}) - f(\vec{a}).$$

For differential forms, this becomes (writing the geometric boundary of C as $\partial C = \{\vec{b}, \vec{a}\}$)

$$\int_C df = \int_{\partial C} f$$

where we define $\displaystyle\int_{\partial C} f \overset{def}{=} f(\vec{b}) - f(\vec{a})$.

Proof. First assume that C is smooth with orientation-preserving parametrization $\vec{r} : [a, b] \to \mathbb{R}^n$. Then we have (by the Chain Rule and the Fundamental Theorem of Calculus)

$$\begin{aligned}\int_C \nabla f \cdot d\vec{s} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \\ &= \int_a^b Df(\vec{r}(t)) D\vec{r}(t)\, dt \\ &= \int_a^b (f \circ \vec{r})'(t)\, dt \\ &= f(\vec{r}(b)) - f(\vec{r}(a)) \\ &= f(\vec{b}) - f(\vec{a}).\end{aligned}$$

If $C = C_1 \cup \cdots \cup C_k$ is piecewise smooth, then denote by $\vec{a}_i$ and $\vec{b}_i$ the starting and ending points of C_i, and note that $\vec{a}_1 = \vec{a}$, $\vec{b}_k = \vec{b}$, and $\vec{b}_i = \vec{a}_{i+1}$ for each $i = 1, \ldots, k-1$. Therefore we have (by the result for smooth curves)

$$\begin{aligned}\int_C \nabla f \cdot d\vec{s} &= \int_{C_k} \nabla f \cdot d\vec{s} + \cdots + \int_{C_1} \nabla f \cdot d\vec{s} \\ &= f(\vec{b}_k) \underbrace{- f(\vec{a}_k) + f(\vec{b}_{k-1})}_{=0} \underbrace{- f(\vec{a}_{k-1}) + f(\vec{b}_{k-2})}_{=0} - \cdots \underbrace{- f(\vec{a}_2) + f(\vec{b}_1)}_{=0} - f(\vec{a}_1) \\ &= f(\vec{b}) - f(\vec{a}).\end{aligned}$$

□

Example 90. Going back to our example above, we note that since $\vec{F} = \nabla f$ on $\mathbb{R}^2$, where $f(x,y) = e^{x^2y^2} + \sin(y)$, and since the staring point of C and the ending point of C are both $(0,0)$, the Fundamental Theorem of Line Integrals gives

$$\int_C \vec{F} \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} = f(0,0) - f(0,0) = 0.$$

This is *sooooooo* much easier than explicitly computing what we labeled (an unholy mess)!

Lecture 23: Conservative Vector Fields and Green's Theorem

Learning Objectives:

- Wield several equivalent characterizations of conservative fields.
- Frame Green's Theorem in terms of vector fields and differential forms.
- Apply Green's Theorem to interchange line integrals in $\mathbb{R}^2$ with double integrals.

The previous example generalizes as follows.

Corollary 1. Let $U \subseteq \mathbb{R}^n$ be open. If $\vec{F} : U \to \mathbb{R}^n$ is continuous and conservative on U, then

$$\oint_C \vec{F} \cdot d\vec{s} = 0$$

for every oriented piecewise-smooth closed curve C in U.

Here the notation $\oint_C \vec{F} \cdot d\vec{s}$ indicates that we are integrating $\vec{F}$ over a *closed* curve.

Proof. Let $f : U \to \mathbb{R}$ be a potential function for $\vec{F}$ on U, and let C be a piecewise-smooth oriented closed curve in U. Let $\vec{a}$ denote the starting point of C (which is also the ending point of C because C is closed). Then the Fundamental Theorem of Line Integrals gives

$$\oint_C \vec{F} \cdot d\vec{s} = \oint_C \nabla f \cdot d\vec{s} = f(\vec{a}) - f(\vec{a}) = 0.$$

□

Perhaps surprisingly, the converse of the previous corollary also holds: If $\vec{F} : U \to \mathbb{R}^n$ is a continuous vector field on U and if $\oint_C \vec{F} \cdot d\vec{s} = 0$ for every piecewise-smooth oriented closed curve C in U, then $\vec{F}$ is conservative on U. These conditions are also equivalent to the statement that the line integral of a conservative vector field depends only on the starting and ending points of the curve, and not on the curve itself. (To capture this, we say that conservative vector fields have *path-independent line integrals.*) These conditions characterize conservative vectors fields, and we collect them in a theorem[21].

[21] Here, connected just means that the set is in "one piece". A crucial fact from topology is that in a connected open subset of $\mathbb{R}^n$, every pair of points can be joined by a smooth curve.

Theorem 14 (Conservative Vector Fields)**.** Let $U \subseteq \mathbb{R}^n$ be open and connected. If $\vec{F} : U \to \mathbb{R}^n$ is continuous, then the following statement are equivalent.

(a) $\vec{F}$ is conservative on U.

(b) $\displaystyle\oint_C \vec{F} \cdot d\vec{s} = 0$ for every oriented piecewise-smooth closed curve C in U.

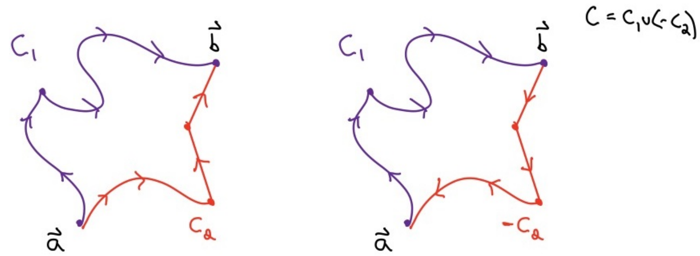
(c) $\vec{F}$ has **path-independent line integrals** in U. That is, if C_1, C_2 are any two oriented piecewise-smooth curves with the same starting and ending points, then

$$\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}.$$

Proof. We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

The implication (a) $\Rightarrow$ (b) was exactly the content of the previous corollary.

Let's now show that (b) $\Rightarrow$ (c). Suppose that (b) holds, and that C_1, C_2 are any two piecewise-smooth oriented curves with the same starting and ending points. Let $-C_2$ denote the piecewise-smooth curve C_2 but with the orientation reversed. Then the starting point of C_1 is the ending point of $-C_2$, and the ending point of C_1 is the starting point of $-C_2$, and therefore $C = C_1 \cup (-C_2)$ is a piecewise-smooth closed curve.



By assumption (b),

$$0 = \oint_C \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{-C_2} \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} - \int_{C_2} \vec{F} \cdot d\vec{s},$$

so that $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$. This proves (c).

The proof that (c) $\Rightarrow$ (a) is accomplished by defining an explicit potential function f for $\vec{F}$. The proof of the general case is no more difficult (but certainly more notationally intense) than the proof of the special case where $n = 2$ and $U = \mathbb{R}^2$, so we will give the proof in that special case. That is we will give the proof in the case where $\vec{F} = P\vec{i} + Q\vec{j}$ is defined on all of $\mathbb{R}^2$. To simplify notation a bit, we will prove the result using the differential form version of $\vec{F}$, $Pdx + Qdy$. Most of the proof will be in your homework, but here is the idea (and part of the proof). Suppose that (c) holds. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x, y) \stackrel{def}{=} \int_C Pdx + Qdy,$$

where C is any piecewise-smooth oriented curve in $\mathbb{R}^2$ that starts at $(0, 0)$ and ends at (x, y). The assumption that $\vec{F}$ has path-independent line integrals ensures that the value of the integral does not depend on C, and therefore $f(x,y)$ is well-defined. The key to showing that the partial derivatives of f at (x,y) exist is to make a "useful choice" of path C connecting $(0,0)$ with (x,y). We'll show that $f_x(x,y) = P(x,y)$, and on your homework you will show that $f_y(x,y) = Q(x,y)$. In both cases, we will need the part of the Fundamental Theorem of Calculus that involves differentiating integrals.

Choose C to be the piecewise-smooth oriented curve consisting first of the oriented line segment C_1 starting at $(0,0)$ and ending at $(0,y)$, followed by the oriented line segment C_2 starting at $(0,y)$ and ending at (x,y). Then we can parametrize C_1 and C_2 (respectively) with

$$\vec{r}_1(t) = (0,ty),\ 0 \le t \le 1, \qquad \vec{r}_2(t) = (tx,y),\ 0 \le t \le 1.$$

It follows that

$$\begin{aligned} f(x,y) &= \int_C P dx + Q dy \\ &= \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy \\ &= \int_0^1 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\, dt + \int_0^1 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\, dt \\ &= \int_0^1 Q(0,ty) y\, dt + \int_0^1 P(tx,y) x\, dt \\ &= \int_0^y Q(0,u)\, du + \int_0^x P(u,y)\, du, \end{aligned}$$

where in the last line we made the substitutions $u = ty$ and $u = tx$ in (respectively) the first and second integrals. Note that $\displaystyle\int_0^y Q(0,t)\, dt$ is constant in x, and that the function $P(t,y)$ (of t) is continuous on $[0,x]$. Therefore the Fundamental Theorem of Calculus implies that $f(x,y)$ is differentiable with respect to x and

$$f_x(x,y) = 0 + P(x,y) = P(x,y).$$

You will complete the argument by computing $f_y(x,y)$ on your homework. □

Example 91. Recall that we showed that the vector field $\vec{F} = \frac{-y}{x^2+y^2}\,\vec{\imath} + \frac{x}{x^2+y^2}\,\vec{\jmath}$ satisfies

$$\int_C \vec{F} \cdot d\vec{s} = 2\pi,$$

where C is the unit circle $x^2+y^2 = 1$ oriented counterclockwise. Because $\vec{F}$ is continuous on $\mathbb{R}^2 - \{(0,0)\}$, the Conservative Vector Fields Theorem then implies that $\vec{F}$ is *not* conservative on $\mathbb{R}^2 - \{(0,0)\}$. That is, there is no C^1 function $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ with $\nabla f = \vec{F}$ on $\mathbb{R}^2 - \{(0,0)\}$! However, on your homework you will show that if (say) we remove the positive x-axis from $\mathbb{R}^2 - \{(0,0)\}$, then it *is* possible to find a potential for $\vec{F}$ on this (slightly smaller) set. Therefore, in the statement that "$\vec{F}$ is conservative on U", the set U can sometimes play just as important a role as $\vec{F}$!

Example 92. Compute $\displaystyle\int_C (1 - 2y + 2xe^{x^2}y^2)dx + (2ye^{x^2} + 2x)dy$, where C is the piecewise-smooth oriented curve consisting of the line segment $(4,0)$ to $(0,0)$, followed by the line segment connecting $(0,0)$ to $(0,-2)$.

A quick test tells us that this vector field is not conservative, since

$$(2ye^{x^2}+2x)_x-(1-2y+2xe^{x^2}y^2)_y = 4xye^{x^2}+2-(-2+4xye^{x^2}) = 4 \neq 0.$$

However, the vector field is *almost conservative*. After all, if the $-2y$ and $2x$ weren't there, then we would have gotten 0 above!

Therefore, we might try splitting the vector field $\vec{F}$ to get

$$\vec{F}(x,y) = [(1+2xe^{x^2}y^2)\vec{i}+2ye^{x^2}\vec{j}]+(-2y\vec{i}+2x\vec{j}).$$

Note that the first part *is* conservative over $\mathbb{R}^2$, since $\mathbb{R}^2$ is simply connected and

$$\operatorname{curl}[(1+2xe^{x^2}y^2)\vec{i}+2ye^{x^2}\vec{j}] = 0$$

at every point $(x,y)\in\mathbb{R}^2$.

Therefore, we know that this first part has a potential function. In this case, we can run through our antidifferentiation argument to get

$$(1+2xe^{x^2}y^2)\vec{i}+2ye^{x^2}\vec{j} = \nabla(x+y^2e^{x^2}),$$

and therefore

$$\begin{aligned}
\int_C (1-2y+2xe^{x^2}y^2)dx+(2ye^{x^2}+2x)dy &= \int_C \nabla(x+y^2e^{x^2})\cdot d\vec{s}+\int_C -2ydx+2xdy \\
&= \underbrace{(0+(-2)^2e^{0^2}-(4-0^2e^{4^2}))}_{=0}+\int_C -2ydx+2xdy \\
&= -\int_0^4 -2(0)+2x(0)dy-\int_{-2}^0 -2y(0)+2(0)dy \\
&= 0.
\end{aligned}$$

In the penultimate line, we split C into its two segments, changed the orientation of each (hence the $-$ sign in front of the integrals), and then parametrized them using $\vec{r}_1(x)=(x,0)$ for $0\le x\le 4$ and $\vec{r}_2(y)=(0,y)$ for $-2\le y\le 0$.

Green's Theorem

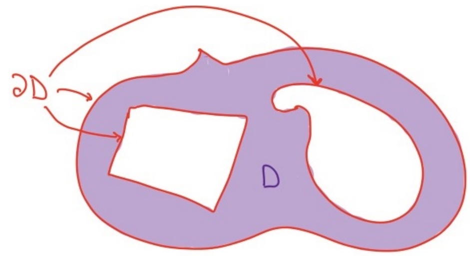
We have now proved one generalization of the Fundamental Theorem of Calculus of the form

$$\int_C df = \int_{\partial C} f,$$

where f is a 0-form (so df is a 1-form) and C is a "one-dimensional manifold" (i.e. a curve) (so that ∂C is a "zero-dimensional manifold" (i.e. a set of points)). For our next generalization (known as *Green's Theorem*), we will go 'up a dimension' and look for formula of the type

$$\iint_D d\omega = \int_{\partial D}\omega,$$

where $D\subset\mathbb{R}^2$ is a region whose boundary ∂D consists of piecewise-smooth curves, and where ω is a 1-form (so that $d\omega$ is a 2-form).



To flesh out what we need to prove, let's note that if $\omega = Pdx + Qdy$, then we would need that

$$\int_{\partial D} Pdx + Qdy = \iint_D d(Pdx + Qdy) = \iint_D (Q_x - P_y)\, dx \wedge dy = \iint_D (Q_x - P_y)\, dA(x, y).$$

Note that if $\vec{F} = P\vec{i} + Q\vec{j}$ is the vector-field represented by $\omega = Pdx + Qdy$, then $d\omega = \text{curl}(\vec{F})\, dx \wedge dy$ (where $\text{curl}(\vec{F})$ is the scalar curl of $\vec{F}$). The orientation of the curves that comprise ∂D is crucial here, as getting the incorrect orientation should cause us to be off by a factor of -1. It is not obvious what this orientation should be right now, but we will see in the proof that there is exactly one possible choice of orientation that is reasonable.

Lecture 24: More Green's Theorem

Learning Objectives:

- Apply Green's Theorem to interchange line integrals in $\mathbb{R}^2$ with double integrals.

Last time we finished class by motivating the structure of Green's Theorem in terms of differential forms. We discussed that Green's Theorem should have the form

$$\iint_D \text{curl}\vec{F}(x,y)\,dA(x,y) = \oint_{\partial D} \vec{F}\cdot d\vec{s} \qquad \sim \qquad \iint_D d\omega = \int_{\partial D} \omega,$$

where

$$\vec{F} = P\vec{i} + Q\vec{j} \sim Pdx + Qdy = \omega$$

and

$$\text{curl}\vec{F}(x,y) = Q_x(x,y) - P_y(x,y) \sim (Q_x - P_y)\,dx\wedge dy = d(Pdx + Qdy).$$

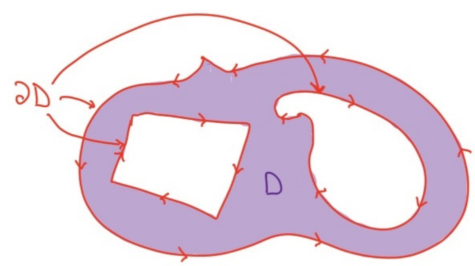
Here is the formal statement.

Theorem 15 (Green's Theorem)**.** Let $D \subset \mathbb{R}^2$ a bounded region such that ∂D consists of a finite union of closed piecewise-smooth curves $C_1, \dots C_k$, where each C_j is oriented so that, while traveling along C_j, the region D is "on the left". If $\vec{F} = P\vec{i} + Q\vec{j}$ is a C^1 vector field on an open set $U \subset \mathbb{R}^2$ with $D \subset U$, then

$$\iint_D \text{curl}\vec{F}(x,y)\,dA(x,y) = \oint_{\partial D} \vec{F}\cdot d\vec{s} \overset{def}{=} \int_{C_1} \vec{F}\cdot d\vec{s} + \cdots + \int_{C_k} \vec{F}\cdot d\vec{s}.$$

In terms of differential forms, this can be expressed as

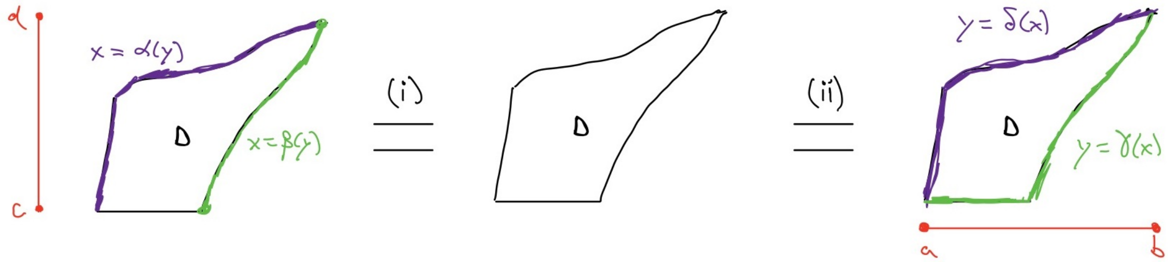
$$\iint_D (Q_x - P_y)\,dx\wedge dy = \iint_D d(Pdx + Qdy) = \int_{\partial D} Pdx + Qdy.$$



Proof. The proof of Green's Theorem has two main steps. First, we establish the theorem in the special case where D is an elementary region that can be written in both forms

$$D \overset{(i)}{=} \{(x,y) \;:\; c \le y \le d,\ \alpha(y) \le x \le \beta(y)\}$$
$$\overset{(ii)}{=} \{(x,y) \;:\; a \le x \le b,\ \gamma(x) \le y \le \delta(x)\},$$

where $\alpha, \beta, \gamma, \delta$ are continuous, (piecewise) C^1 functions.



The second step is to show that a general region D that satisfies the hypotheses of the theorem can be split into a finite number of these special elementary regions, and that when we split up $\iint_D (Q_x - P_y)\, dA(x,y)$ into a sum of integrals of these regions, apply the special case to each piece, and then add up the resulting line integrals, we obtain the expected integral over ∂D. The details of this second part are very technical (of the type that belong in a course in real analysis), but we will give an argument in a concrete example to show how the process works in general.

Step 1: Proof when D has form (i) and (ii). To make the details of the proof easier to write down, let's assume that $\alpha, \beta, \gamma, \delta$ are actually C^1 functions. We split up $\iint_D \text{curl}\vec{F}(x,y)\, dA(x,y)$ using the linearity of the integral, and then apply Fubini's Theorem differently in each piece to obtain

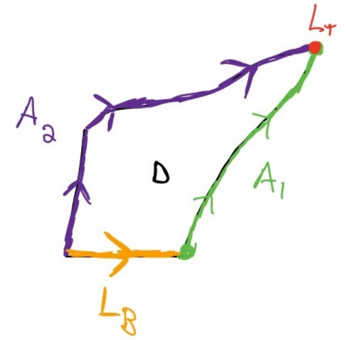
$$\begin{aligned}
&\iint\limits_D \text{curl}(\vec{F})(x,y)\, dA(x,y) \\
&\qquad = \iint\limits_D Q_x(x,y)\, dA(x,y) + \iint\limits_D -P_y(x,y)\, dA(x,y) \\
&\qquad = \int_c^d \int_{\alpha(y)}^{\beta(y)} Q_x(x,y)\, dxdy + \int_a^b \int_{\gamma(x)}^{\delta(x)} -P_y(x,y)\, dydx \\
&\qquad = \int_c^d \Big[Q(x,y) \Big]_{\alpha(y)}^{\beta(y)} dy + \int_a^b \Big[-P(x,y) \Big]_{\gamma(x)}^{\delta(x)} dx \\
&\qquad = \int_c^d (Q(\beta(y),y) - Q(\alpha(y),y))\, dy + \int_a^b (P(x,\gamma(x)) - P(x,\delta(x)))\, dx \\
&\qquad = \int_c^d Q(\beta(y),y)\, dy - \int_c^d Q(\alpha(y),y))\, dy + \int_a^b (P(x,\gamma(x))\, dx - \int_a^b P(x,\delta(x)))\, dx
\end{aligned}$$

On the other hand, consider that (using representation (i) of D) we can represent $\partial D = A_1 \cup (-L_T) \cup (-A_2) \cup L_B$, where A_1 and A_2 are the (piecewise-smooth) curves parametrized by

$$\vec{r}_1(y) = (\beta(y), y), \quad c \le y \le d \qquad \text{and} \qquad \vec{r}_2(y) = (\alpha(y), y), \quad c \le y \le d,$$

and L_T and L_B are the (possibly absent if $\alpha(c) = \beta(c)$ or $\alpha(d) = \beta(d)$) line segments parametrized by

$$\vec{\ell}_T(t) = (t, d),\ \alpha(d) \le t \le \beta(d), \qquad \vec{\ell}_B(t) = (t, c),\ \alpha(c) \le t \le \beta(c).$$



In terms of these parametrizations, we have

$$\int_c^d Q(\beta(y),y)\,dy = \int_c^d (Q(\beta(y),y)\vec{j})\cdot(\beta'(y)\vec{i}+1\vec{j})\,dy = \int_c^d (Q(\vec{r}_1(y))\vec{i})\cdot\vec{r}_1{}'(y)\,dy = \int_{A_1}(Q\vec{j})\cdot d\vec{s}$$

and, similarly,

$$-\int_c^d Q(\alpha(y),y)\,dy = -\int_{A_2}(Q\vec{j})\cdot d\vec{s} = \int_{-A_2}(Q\vec{j})\cdot d\vec{s}.$$

Since $(Q(\vec{\ell}_T(t))\vec{j})\cdot\vec{\ell}_T{}'(t) = (Q(\vec{\ell}_T(t))\vec{j})\cdot\vec{i} = 0$ for all t, $\int_{L_T}(Q\vec{j})\cdot d\vec{s} = 0$. Similarly, $\int_{L_B}(Q\vec{j})\cdot d\vec{s} = 0$.

Therefore the first pair of integrals can be written as

$$\begin{aligned}\int_c^d Q(\beta(y),y)\,dy - \int_c^d Q(\alpha(y),y))\,dy \\ &= \int_{A_1}(Q\vec{j})\cdot d\vec{s} + \underbrace{\int_{-L_T}(Q\vec{j})\cdot d\vec{s}}_{0} + \int_{-A_2}(Q\vec{j})\cdot d\vec{s} + \underbrace{\int_{L_B}(Q\vec{j})\cdot d\vec{s}}_{0} \\ &= \oint_{\partial D}(Q\vec{j})\cdot d\vec{s}.\end{aligned}$$

Repeating this argument using representation (ii) of D, and representing $\partial D = B_1\cup L_R\cup(-B_2)\cup(-L_L)$, where B_1 and B_2 are curves parametrized by

$$\vec{p}_1(x) = (x,\gamma(x)),\quad a\le x\le b \qquad\text{and}\qquad \vec{p}_2(x) = (x,\delta(x)),\quad a\le x\le b$$

and L_R and L_L are the (possibly absent if $\gamma(a)=\delta(a)$ or $\gamma(b)=\delta(b)$) line segments parametrized by

$$\vec{\ell}_R(t) = (b,t),\ \gamma(b)\le t\le\delta(b), \qquad \vec{\ell}_L(t) = (a,t),\ \gamma(a)\le t\le\delta(a),$$

leads to

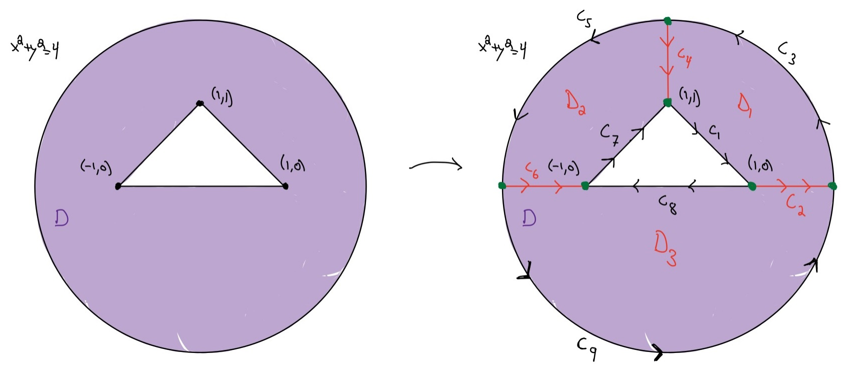
$$\begin{aligned}\int_a^b (P(x,\gamma(x))\,dx - \int_a^b P(x,\delta(x)))\,dx \\ &= \int_{B_1}(P\vec{i})\cdot d\vec{s} + \underbrace{\int_{L_R}(P\vec{i})\cdot d\vec{s}}_{0} - \int_{B_2}(P\vec{i})\cdot d\vec{s} - \underbrace{\int_{L_L}(P\vec{i})\cdot d\vec{s}}_{0} \\ &= \int_{B_1}(P\vec{i})\cdot d\vec{s} + \int_{L_R}(P\vec{i})\cdot d\vec{s} + \int_{-B_2}(P\vec{i})\cdot d\vec{s} + \int_{-L_L}(P\vec{i})\cdot d\vec{s} \\ &= \oint_{\partial D}(P\vec{i})\cdot d\vec{s}.\end{aligned}$$

Therefore we have

$$\iint_D \operatorname{curl}\vec{F}(x,y)\,dA(x,y) = \oint_{\partial D} (Q\vec{j})\cdot d\vec{s} + \oint_{\partial D} (P\vec{i})\cdot d\vec{s} = \oint_{\partial D} (P\vec{i} + Q\vec{j})\cdot d\vec{s},$$

where the piecewise-smooth closed curve ∂D is parametrized so that D is "on the left" when traversing ∂D (in this special case, this means that ∂D is parametrized in the counter-clockwise direction).

Step 2: Illustration of how to handle general regions D. One can show that every D satisfying the hypotheses of Green's Theorem can be split up into a finite number of regions that satisfy the hypotheses of Step 1. For example, the region D that is enclosed by the circle $x^2 + y^2 = 4$ and lies exterior to the triangle with vertices $(\pm 1, 0)$ and $(0, 1)$ can be split into $D = D_1 \cup D_2 \cup D_3$ as follows:



With the oriented curves labeled in the picture above, we have (applying the special case to each of D_1, D_2, and D_3),

$$\begin{aligned}
\iint_D \operatorname{curl}\vec{F}(x,y)\,dA &= \iint_{D_1} \operatorname{curl}\vec{F}(x,y)\,dA + \iint_{D_2} \operatorname{curl}\vec{F}(x,y)\,dA + \iint_{D_3} \operatorname{curl}\vec{F}(x,y)\,dA \\
&= \oint_{\partial D_1} \vec{F}\cdot d\vec{s} + \oint_{\partial D_2} \vec{F}\cdot d\vec{s} + \oint_{\partial D_3} \vec{F}\cdot d\vec{s} \\
&= \int_{C_1} \vec{F}\cdot d\vec{s} + \int_{C_2} \vec{F}\cdot d\vec{s} + \int_{C_3} \vec{F}\cdot d\vec{s} + \int_{C_4} \vec{F}\cdot d\vec{s} \\
&\quad + \int_{C_5} \vec{F}\cdot d\vec{s} + \int_{C_6} \vec{F}\cdot d\vec{s} + \int_{C_7} \vec{F}\cdot d\vec{s} + \underbrace{\int_{-C_4} \vec{F}\cdot d\vec{s}}_{-\int_{C_4} \vec{F}\cdot d\vec{s}} \\
&\quad + \int_{C_8} \vec{F}\cdot d\vec{s} + \underbrace{\int_{-C_6} \vec{F}\cdot d\vec{s}}_{-\int_{C_6} \vec{F}\cdot d\vec{s}} + \int_{C_9} \vec{F}\cdot d\vec{s} + \underbrace{\int_{-C_2} \vec{F}\cdot d\vec{s}}_{-\int_{C_2} \vec{F}\cdot d\vec{s}} \\
&= \int_{C_1} \vec{F}\cdot d\vec{s} + \int_{C_8} \vec{F}\cdot d\vec{s} + \int_{C_7} \vec{F}\cdot d\vec{s} + \int_{C_3} \vec{F}\cdot d\vec{s} + \int_{C_5} \vec{F}\cdot d\vec{s} + \int_{C_9} \vec{F}\cdot d\vec{s} \\
&= \oint_{C_1\cup C_8\cup C_7} \vec{F}\cdot d\vec{s} + \oint_{C_3\cup C_5\cup C_9} \vec{F}\cdot d\vec{s} \\
&= \oint_{\partial D} \vec{F}\cdot d\vec{s}.
\end{aligned}$$

Note that the portion of ∂D that consists of the circle $x^2 + y^2 = 4$ is parametrized in the counterclockwise direction, and the portion of ∂D that consists of the triangle with vertices $(\pm 1, 0)$ and $(0, 1)$ is parametrized in the *clockwise* direction. In each case, the region D is "on the left" as one traverses the curve. Note that all of the terms in our computation that arose from integration along curves that were not part of ∂D canceled out. □

Remark 47. For a physical interpretation of Green's Theorem, we must turn to scalar curl.

Recall that the line integral $\oint_{\partial D} Pdx + Qdy$ measures the circulation of the vector field $\vec{F} = P\vec{i} + Q\vec{j}$ around the boundary of D (with positive circulation corresponding to 'net positive rotation' along ∂D in the direction of orientation). On the other hand, $\text{curl}\vec{F}(x, y) = Q_x(x, y) - P_y(x, y)$ measures the counterclockwise "twisting" of $\vec{F}$ at the point (x, y) (with negative values giving clockwise twisting). Therefore the conclusion

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \text{curl}(\vec{F})\, dA$$

or Green's Theorem indicates that we can detect the total net amount of "counterclockwise twisting" of $\vec{F}$ throughout D (measured by the double integral) by computing the net "counterclockwise rotation" of $\vec{F}$ along the ∂D.

Remark 48. The physical interpretation of Green's Theorem in the previous remark assumes our intuition for what the (scalar) curl of $\vec{F} = P\vec{i} + Q\vec{j}$ means. In particular, while we used our "intuition" that the scalar curl $\text{curl}\vec{F}(x, y) = Q_y(x, y) - P_x(x, y)$ of a C^1 vector field $\vec{F} = P\vec{i} + Q\vec{j}$ on $\mathbb{R}^2$ somehow represents the "rotation" or "twisting" of $\vec{F}$ at a point, we can actually use Green's Theorem to *justify* this intuition. By Exercise 4 on Homework 4 (and Green's Theorem) we have (at a point $(x_0, y_0) \in \mathbb{R}^2$),

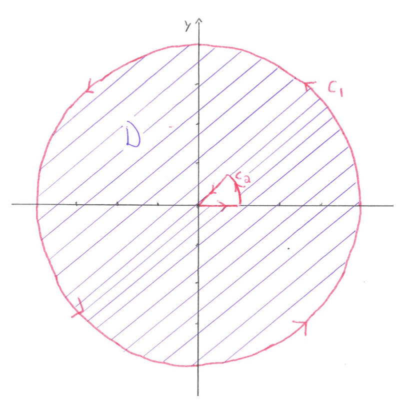
$$\text{curl}\vec{F}(x_0, y_0) = \lim_{r \to 0+} \frac{1}{\pi r^2} \iint_{B_r(x_0, y_0)} \text{curl}\vec{F}(x, y)\, dA(x, y) = \lim_{r \to 0+} \frac{1}{\pi r^2} \oint_{C_r} \vec{F} \cdot d\vec{s},$$

where C_r is the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ oriented in the counterclockwise direction. Therefore $\text{curl}\vec{F}(x_0, y_0)$ does indeed measure "infinitesimal counterclockwise rotation" of $\vec{F}$ at (x_0, y_0).

Applications and Examples

Example 93. Let C_1 be the circle $x^2 + y^2 = 16$, and C_2 be the boundary of the region in the first quadrant which is enclosed by $x^2 + y^2 = 1$ and bounded above by $x = y$. Here we assume that C_1 and C_2 are both oriented in the counterclockwise direction. Compute

$$\oint_{C_1} -ydx + xdy - \oint_{C_2} -ydx + xdy.$$



To do this, let D be the region between the two curves. Then the boundary of D consists of C_1 and C_2, but in order to apply Green's Theorem we need C_2 oriented in the clockwise direction, and for the difference above to be a sum. We can kill both birds with one stone by reversing the orientation of C_2 (and thereby picking up an additional factor of -1 in front of the second integral):

$$\begin{aligned} & \int_{C_1} -ydx + xdy - \int_{C_2} -ydx + xdy \\ = & \int_{C_1} -ydx + xdy + \int_{-C_2} -ydx + xdy \\ = & \oint_{\partial D} -ydx + xdy \\ = & \iint_D [(x)_x - (-y)_y] dA(x,y) \\ = & \iint_D 2\, dA(x,y) \\ = & \ 2\text{Area}(D), \end{aligned}$$

where we applied Green's theorem in the antepenultimate step. Since D is merely the disc $x^2 + y^2 \le 16$ with $\frac{1}{8}$ of the disc $x^2 + y^2 \le 1$ removed, we can compute this last integral to be

$$\oint_{\partial D} -ydx + xdy = 2\text{Area}(D) = 2(\pi 4^2 - \frac{1}{8}\pi 1^2) = \frac{127\pi}{4}.$$

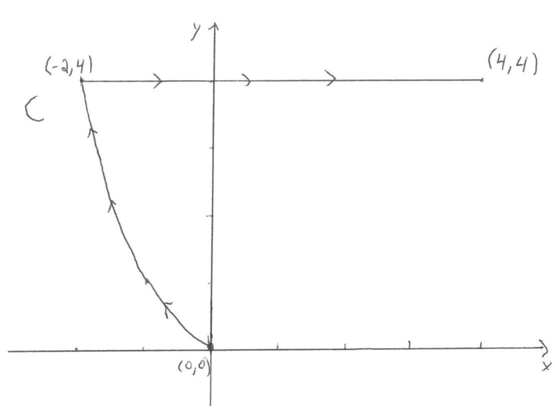
As the last line of the previous example suggests, there is nothing special about the region D. Indeed, Green's theorem immediately gives us the very interesting result that the area of a region can be found by computing a line integral over the boundary!

Corollary 2. If $D \subset \mathbb{R}^2$ is a region satisfying the hypotheses of Green's Theorem, and if each piecewise-smooth closed curve forming ∂D is oriented so that D is on the left, then

$$\text{Area}(D) = \frac{1}{2} \oint_{\partial D} -ydx + xdy.$$

Example 94 (Closing off a curve)**.** Let C be the oriented piecewise-smooth curve consisting of the piece of the parabola $y = x^2$ connecting $(0,0)$ to $(-2,4)$, followed by the line segment connecting $(-2,4)$ to $(4,4)$. Compute

$$\int_C (y^2 - x^3 + x^2 e^{x\cos(x)})dx - y^2 e^{y\cos(y)} dy.$$

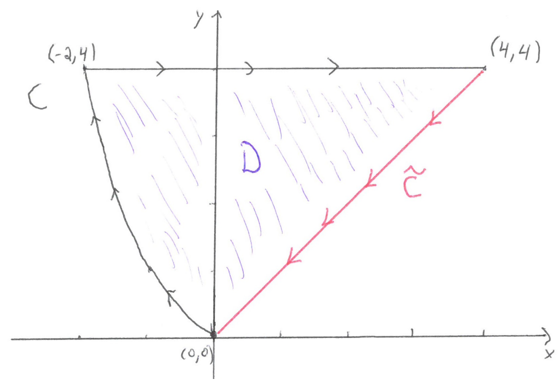


This problem seems intractible, since we have very little hope of computing any antiderivatives involving the terms $x^2e^{x\cos(x)}$ and $y^2e^{y\cos(y)}$. Indeed, if we use the parameterization $(x,y)=(x,4)$, $-2\le x\le 4$ for the line segment, then that piece of the line integral is

$$\int_{-2}^{4} 16-x^3+x^2e^{x\cos(x)}dx,$$

which we have absolutely no hope of evaluating!

To get around this, let's 'close off' this curve with the line segment $\tilde{C}$ connecting $(4,4)$ to $(0,0)$:



Our integral then becomes

$$\begin{aligned}
&\int_C (y^2-x^3+x^2e^{x\cos(x)})dx-y^2e^{y\cos(y)}dy\\
=\;&\int_C (y^2-x^3+x^2e^{x\cos(x)})dx-y^2e^{y\cos(y)}dy+\int_{\tilde{C}}(y^2-x^3+x^2e^{x\cos(x)})dx-y^2e^{y\cos(y)}dy\\
&\quad-\int_{\tilde{C}}(y^2-x^3+x^2e^{x\cos(x)})dx-y^2e^{y\cos(y)}dy\\
=\;&\oint_{C+\tilde{C}}(y^2-x^3+x^2e^{x\cos(x)})dx-y^2e^{y\cos(y)}dy+\int_{-\tilde{C}}(y^2-x^3+x^2e^{x\cos(x)})dx-y^2e^{y\cos(y)}dy,
\end{aligned}$$

where in the last step we reversed the orientation of the second integral, replacing $\tilde{C}$ with $-\tilde{C}$ (and thereby canceling the -1 in front of the integral).

The simple closed curve $C+\tilde{C}$ is the boundary of the region D pictured. Its orientation does not allow us to directly apply Green's Theorem, but if we change the orientation of the curve we will have what we need. Indeed,

$$\begin{aligned}
&\oint_{C+\tilde{C}} (y^2 - x^3 + x^2 e^{x\cos(x)})dx - y^2 e^{y\cos(y)}dy \\
&= -\oint_{-(C+\tilde{C})} (y^2 - x^3 + x^2 e^{x\cos(x)})dx - y^2 e^{y\cos(y)}dy \\
&= -\oint_{\mathrm{b}D} (y^2 - x^3 + x^2 e^{x\cos(x)})dx - y^2 e^{y\cos(y)}dy \\
&= -\iint_D [(-y^2 e^{y\cos(y)})_x - (y^2 - x^3 + x^2 e^{x\cos(x)})_y] dA(x,y) \\
&= -\iint_D -2y dA(x,y) \\
&= \int_0^4 \int_{-\sqrt{y}}^{y} 2y dx dy \\
&= \int_0^4 2y^2 + 2y^{\frac{3}{2}} dy \\
&= \frac{256}{15}.
\end{aligned}$$

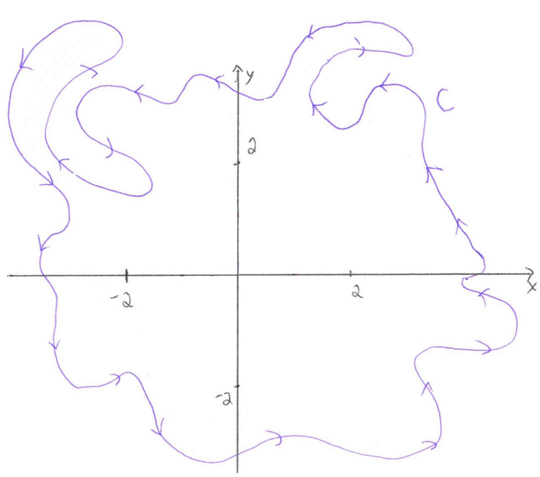
To evaluate our 'leftover' line integral, let's parameterize $-\tilde{C}$ (the line segment connecting $(0,0)$ to $(4,4)$) with $(x,y) = (t,t)$, $0 \le t \le 4$. We therefore have

$$\begin{aligned}
\int_{-\tilde{C}} (y^2 - x^3 + x^2 e^{x\cos(x)})dx - y^2 e^{y\cos(y)}dy &= \int_0^4 (t^2 - t^3 + t^2 e^{t\cos(t)})dt - t^2 e^{t\cos(t)}dt \\
&= \int_0^4 t^2 - t^3 dt = \frac{64}{3} - 64 = -\frac{128}{3}.
\end{aligned}$$

In total,

$$\int_C (y^2 - x^3 + x^2 e^{x\cos(x)})dx - y^2 e^{y\cos(y)}dy = \frac{256}{15} - \frac{128}{3} = -\frac{128}{5}.$$

Example 95 (Replacing a Path)**.** For the vector field $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$ (which is C^1 at every point except for $(0,0)$), we compute $\oint_C \vec{F} \cdot d\vec{s}$, where C is the curve pictured below:



This curve is rather wild, and we would therefore like to try to apply Green's Theorem (to avoid parametrizing such a monster!). However, because the vector field is not C^1 (or even defined!) at $(0,0)$ (which lies in the region enclosed by C), we are not able to apply Green's Theorem. However, if we *could* apply Green's Theorem, then the integrand of the double integral would be (except at $(0,0)$)

$$\text{curl}\vec{F}(x,y) = \left(\frac{x}{x^2+y^2}\right)_x - \left(\frac{-y}{x^2+y^2}\right)_y = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0.$$

Let's pause and remind ourselves that the reason why we wanted to apply Green's Theorem was to avoid computing the line integral of $\vec{F}$ around the curve C. One great corollary of the well-known maxim

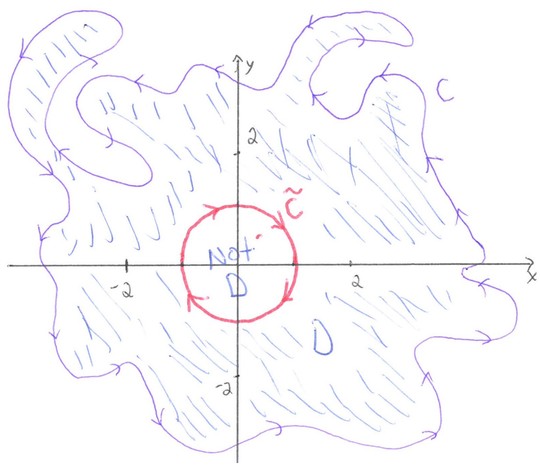
Ask and ye shall receive, seek and ye shall find.

is

If ye ask but do not receive, and seek but do not find, then ask for less and seek harder.

In other words, it may be that we can get by with a bit less here. After all, if $\tilde{C}$ is the unit circle $x^2+y^2=1$, then computing the integral of $\vec{F}$ along $\tilde{C}$ is relatively straightforward (since $\tilde{C}$ is 'nice', and the terms x^2+y^2 in the vector field will just get replaced by 1 when we parametrize)!

Therefore, let's think of D as being the region between C and $\tilde{C}$:



We would like to apply Green's Theorem to D, but we first have to decide on an the correct orientation for $\tilde{C}$. From the picture, it is clear that we need to give $\tilde{C}$ the clockwise orientation (so that D is 'on the left').

Therefore, we will repeat our technique from the previous example to write

$$
\begin{aligned}
\oint_C \vec{F} \cdot d\vec{s} &= \underbrace{\oint_C \vec{F} \cdot d\vec{s} + \oint_{\tilde{C}} \vec{F} \cdot d\vec{s}}_{=\oint_{\partial D} \vec{F} \cdot d\vec{s}} - \oint_{\tilde{C}} \vec{F} \cdot d\vec{s} \\
&= \iint_D \operatorname{curl}\vec{F}(x,y)\, dA(x,y) - \oint_{\tilde{C}} \vec{F} \cdot d\vec{s} \\
&= \iint_D 0\, dA(x,y) - \oint_{\tilde{C}} \vec{F} \cdot d\vec{s} \\
&= -\oint_{\tilde{C}} \vec{F} \cdot d\vec{s} \\
&= \oint_{-\tilde{C}} \vec{F} \cdot d\vec{s}.
\end{aligned}
$$

Since $-\tilde{C}$ is just the circle $x^2 + y^2 = 1$ oriented in the counterclockwise direction, we know from Example 88 that

$$
\oint_C \vec{F} \cdot d\vec{s} = \oint_{-\tilde{C}} \vec{F} \cdot d\vec{s} = 2\pi.
$$

Lecture 25: Vector Surface Integrals

Learning Objectives:

- Describe a compatible orientation of a piecewise-smooth surface in $\mathbb{R}^3$.
- Motivate the notion of vector surface integral in terms of pullbacks of differential forms.
- Interpret vector surface integrals in terms of flux.
- Examine the effects of orientation on the value of a vector surface integral.
- Compute a vector surface integral.

As we develop a notion of integration for vector fields over surfaces, let's recall (and build on) some definitions.

Oriented Smooth and Piecewise-Smooth Surfaces

Definition 33. An **orientation** of a smooth surface $S \subset \mathbb{R}^3$ is a continuous (except at possibly finitely many point) choice of unit normal vectors $\vec{n}$ on S.
If S is a smooth oriented surface, then we say a parametrization $\vec{X}(s,t)$ of S is **orientation-preserving** if $N_{\vec{X}}$ points in the same direction as $\vec{n}$ (except possibly at finitely many points on S). If $N_{\vec{X}}$ points in the opposite direction as $\vec{n}$, then we say that $\vec{X}$ is **orientation-reversing**.

The orientation of a smooth surface designates which direction is "up" from the surface.

Example 96. Note that the upper half S of the cone $z^2 = x^2 + y^2$ is technically a smooth surface, despite having a 'sharp point' at $(0,0,0)$. To see why, consider the parametrization

$$\vec{X}((x^2+y^2)x, (x^2+y^2)y, (x^2+y^2)^{3/2}), \quad (x,y) \in \mathbb{R}^2.$$

One can show that

$$N_{\vec{X}}(x,y) = \begin{bmatrix} -3x(x^2+y^2)^{3/2} \\ -3y(x^2+y^2)^{3/2} \\ 3(x^2+y^2)^2 \end{bmatrix}, \quad \text{so} \quad \|N_{\vec{X}}(x,y)\| = 3\sqrt{2}(x^2+y^2)^2 \neq 0$$

for every $(x,y) \in \mathbb{R}^2$ except $(x,y) = (0,0)$.

Note also that since the k-component of the vector $N_{\vec{X}}(x,y)$ is positive, $N_{\vec{X}}(x,y)$ points "upwards" at each point (at an angle, of course). If we had oriented S with "upward pointing" normal vectors, then $\vec{X}$ would be an orientation-preserving parametrization. If we had oriented S with "downward-pointing" normal vectors, then $\vec{X}$ would be an orientation-reversing parametrization.

We also have a notion of piecewise-smooth oriented surface. The definition is slightly more complicated than that of piecewise-smooth curve, but we will simplify one key detail of the definition to make it more manageable.

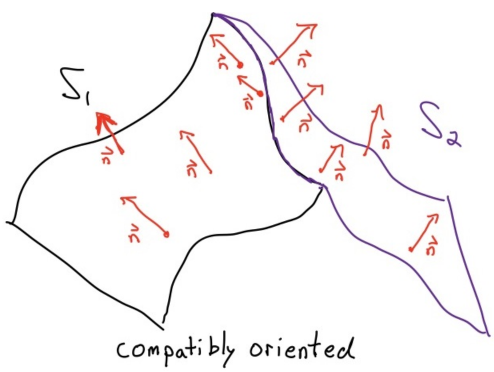
Definition 34. A surface S is **piecewise-smooth** if there are smooth oriented surfaces $S_1, \ldots, S_k$ such that

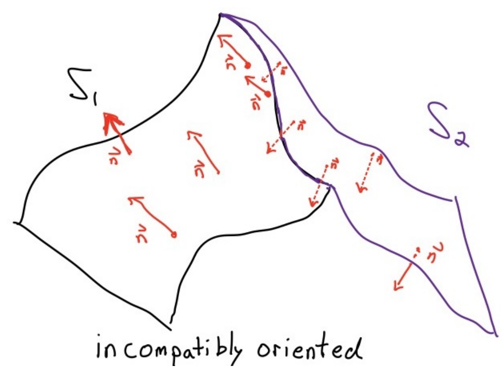
(i) $S = S_1 \cup \cdots \cup S_k$

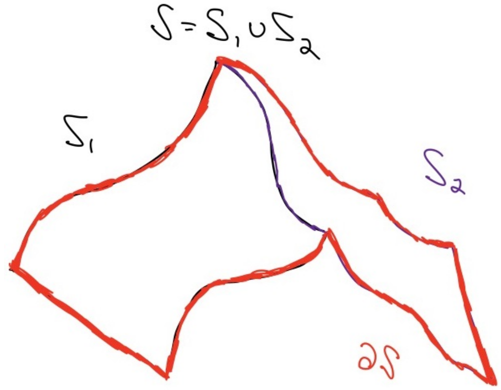
(ii) Any two of $S_1, \ldots, S_k$ intersect only possibly on their geometric boundaries.

(iii) The surfaces $S_1, \ldots, S_k$ are oriented **compatibly**: When $S_i \cap S_j$ is a curve, near any $\vec{p} \in S_i \cap S_j$ the normal vectors $\vec{n}_i$ and $\vec{n}_j$ of S_i and S_j each point towards the "same side" of $S_i \cup S_j$.

The **orientation** on S is taken on each S_i to be the orientation on S_i (but might be undefined when the surfaces $S_1, \ldots, S_k$ intersect). The **(geometric) boundary** of S consists of the portions of the boundaries of $S_1, \ldots, S_k$ that do not overlap.







Remark 49. Note that if $S = S_1 \cup \cdots \cup S_k$ is an oriented piecewise-smooth surface, then we can reverse the orientation on S by reversing the orientation on each of $S_1, \ldots, S_k$. Using $-$ to denote a surface with its orientation reversed, this would mean that $-S = (-S_1) \cup \cdots \cup (-S_k)$.

Vector Surface Integrals

Suppose that $S \subseteq \mathbb{R}^3$ is a smooth oriented surface with unit normal vector $\vec{n}$, and that $\vec{F}$ is a continuous vector field defined on an open set $E \subseteq \mathbb{R}^3$ containing S. In order to determine what should be the 'natural' definition for the "vector surface integral" of $\vec{F}$ over S, we again turn to differential forms. Recall that we considered $\vec{F}$ as equivalent to the 2-form ω given by

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k} \qquad \sim \qquad \omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$$

The "vector surface integral" of $\vec{F}$ over S should agree with the integral of ω over S, which we define in terms of an orientation-*preserving* parametrization

$$\vec{X} : D \to S, \quad \vec{X}(x(s,t), y(s,t), z(s,t))$$

(where $D \subseteq \mathbb{R}^2$ is an elementary region) as $\iint\limits_S \omega \overset{def}{=} \iint\limits_D \vec{X}^*\omega$. (Note that this is *exactly* what we did for integrals of 1-forms over oriented curves, but now for 2-forms over oriented surfaces.) Then, using a computation you performed in Discussion 5, we have

$$\begin{aligned}
\iint\limits_S \omega &\overset{def}{=} \iint\limits_D \vec{X}^*\omega \\
&= \iint\limits_D \Big(P(\vec{X}(s,t))(y_s z_t - z_s y_t) + Q(\vec{X}(s,t))(z_s x_t - x_s z_t) + R(\vec{X}(s,t))(x_s y_t - y_s x_t) \Big)\, ds \wedge dt \\
&= \iint\limits_D \Big(\vec{F}(\vec{X}(s,t)) \cdot N_{\vec{X}}(s,t) \Big)\, ds \wedge dt \\
&= \iint\limits_D \vec{F}(\vec{X}(s,t)) \cdot N_{\vec{X}}(s,t)\, dA(s,t) \\
&= \iint\limits_D \vec{F}(\vec{X}(s,t)) \cdot \underbrace{\left(\frac{1}{\|N_{\vec{X}}(s,t)\|} N_{\vec{X}}(s,t) \right)}_{=\vec{n}(\vec{X}(s,t))} \|N_{\vec{X}}(s,t)\|\, dA(s,t) \\
&= \iint\limits_S \vec{F} \cdot \vec{n}\, dS.
\end{aligned}$$

In other words, $\iint_S \omega$ can be viewed as the *scalar surface integral* of $\vec{F} \cdot \vec{n}$ over S, where $\vec{n}$ is the unit normal vector that gives the orientation of S. Therefore we make the following definition.

Definition 35. Let $\vec{F}$ be a continuous vector field on an open set $E \subseteq \mathbb{R}^3$, and suppose that S is a smooth oriented surface in E with unit normal vectors $\vec{n}$. Then we define the **vector surface integral** of $\vec{F}$ over S to be

$$\iint\limits_S \vec{F} \cdot d\vec{S} \overset{def}{=} \iint\limits_S \vec{F} \cdot \vec{n}\, dS.$$

If $S = S_1 \cup \cdots \cup S_k$ is a piecewise-smooth oriented surface, then we define

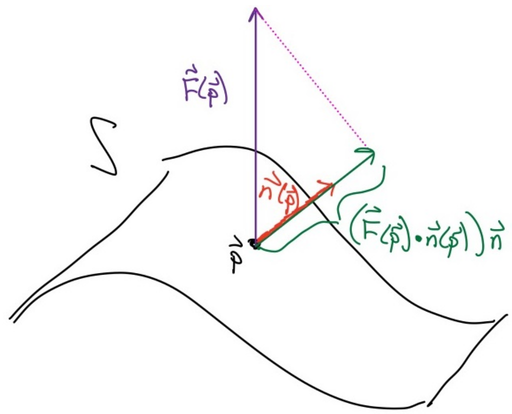
$$\iint\limits_S \vec{F} \cdot d\vec{S} \overset{def}{=} \iint\limits_{S_1} \vec{F} \cdot d\vec{S} + \cdots + \iint\limits_{S_k} \vec{F} \cdot d\vec{S}.$$

Remark 50. The notation $\cdot d\vec{S}$ is an abbreviation of $\cdot \vec{n}\, dS$.

Remark 51. The physical interpretation of a vector surface integral $\iint_S \vec{F} \cdot d\vec{S}$ is quite nice. Suppose that $\vec{F}$ represents the flow of a fluid. At a point $\vec{p} \in S$, $\vec{F}(\vec{p}) \cdot \vec{n}(\vec{p})$ is a scalar that represents the "amount of $\vec{F}$ in the direction of $\vec{n}$". Indeed, to justify this note that

$$\text{proj}_{\vec{n}(\vec{p})} \vec{F}(\vec{p}) = (\vec{F}(\vec{p}) \cdot \vec{n}(\vec{p}))\vec{n}(\vec{p})$$

since $\vec{n}(\vec{p})$ is a unit vector.

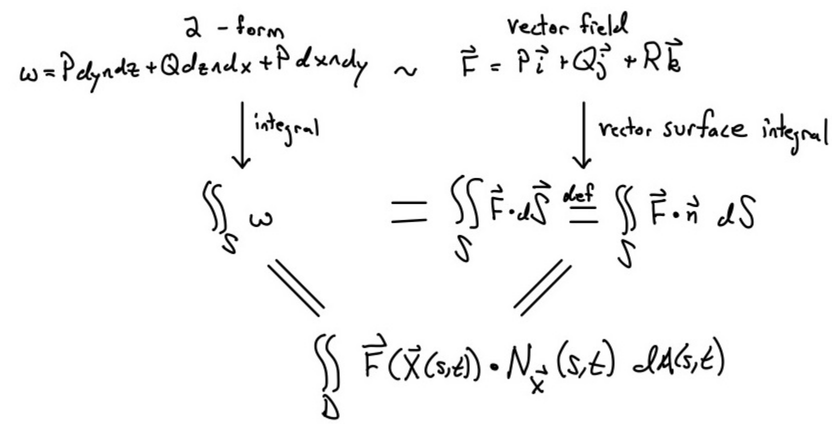


In other words, $\vec{F}(\vec{p}) \cdot \vec{n}(\vec{p})$ represents a measurement of **flux** (or **flow**) **of** $\vec{F}$ **through** S **at** $\vec{p}$ (in the direction given by the orientation of S). When we compute the scalar surface integral

$$\iint\limits_S \vec{F} \cdot \vec{n}\, dS,$$

we are computing the **total (net) flux of** $\vec{F}$ **through** S (in the direction given by the orientation of S).

Remark 52. To summarize, for an oriented smooth surface S in $\mathbb{R}^3$ and a smooth *orientation-preserving* parametrization $\vec{X} : D \to S$ of S, we have the following correspondence:



Remark 53. Note that if S is an orientated smooth surface and $\vec{F}$ a continuous vector field on S, then if $\vec{X} : D \to S$ is an orientation-reversing parametrization we have $\vec{n}(\vec{X}(s,t)) = -\frac{1}{\|N_{\vec{X}}(s,t)\|} N_{\vec{X}}(s,t)$ at each point, so that

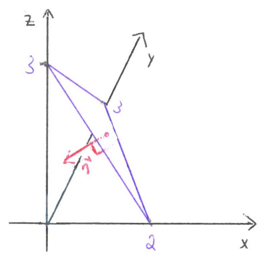
$$\begin{aligned}
\iint\limits_D \vec{F}(\vec{X}(s,t)) \cdot N_{\vec{X}}(s,t)\, dA(s,t) &= -\iint\limits_D \vec{F}(\vec{X}(s,t)) \cdot (-N_{\vec{X}}(s,t))\, dA(s,t) \\
&= -\iint\limits_D \left(\vec{F}(\vec{X}(s,t)) \cdot \vec{n}(\vec{X}(s,t))\right) \|N_{\vec{X}}(s,t)\|\, dA(s,t) \\
&= -\iint\limits_S \vec{F} \cdot \vec{n}\, dS \\
&= -\iint\limits_S \vec{F} \cdot d\vec{S}.
\end{aligned}$$

In other words, *using an orientation-reversing parametrization of S will result in your final answer being off by a factor of* -1. Or, put another way, reversing the orientation of S multiplies the vector line integral of $\vec{F}$ over S by a factor of -1 (just as for vector line integrals)! Or, put a third way, by reversing the orientation of S we are reversing the direction in which we are measuring the flux of $\vec{F}$ through S (and therefore we should end up with the same total net flux, but multiplied by -1).

Examples

Example 97. Let S be the portion of the plane $3x+2y+2z=6$ in the first octant, oriented downward, and let $\vec{F}=4x\vec{i}+x\vec{j}+x\vec{k}$. Without evaluating the integral, determine whether $\iint_S \vec{F}\cdot d\vec{S}$ is positive, negative, or 0.

We sketch the surface below:



Note that at each point on S, $x>0$, and therefore $\vec{F}$ points 'up' from the surface. Hence, $\vec{F}\cdot\vec{n}<0$ at each point (except on the edge where $x=0$), and therefore $\iint_S \vec{F}\cdot d\vec{S}<0$.

Example 98. Compute $\iint_S \vec{F}\cdot d\vec{S}$ from the previous example.

Let's use the following parametrization:

$$\vec{X}(x,y)=\left(x,y,3-y-\frac{3}{2}x\right),\quad 0\le x\le 2,\ 0\le y\le 3-\frac{3}{2}x.$$

For this parametrization,

$$\vec{X}_x(x,y)=\begin{bmatrix}1\\0\\-\frac{3}{2}\end{bmatrix}\quad\text{and}\quad \vec{X}_y(x,y)=\begin{bmatrix}0\\1\\-1\end{bmatrix},\ \text{so that}\ \vec{N}(x,y)=\begin{bmatrix}\frac{3}{2}\\1\\1\end{bmatrix}.$$

Note that since the z-coodinate of this vector is positive, this is actually the *upward*-pointing normal vector to S, which means that we have actually parametrized $-S$ instead of S! No worry; we will just switch the orientation of the surface S (at the cost of a -1 in front of the integral):

$$\begin{aligned}
\iint_S \vec{F}\cdot d\vec{S} &= -\iint_{-S} \vec{F}\cdot d\vec{S} \\
&= -\int_0^2\int_0^{3-\frac{3}{2}x} \begin{bmatrix}4x\\ x\\ x\end{bmatrix}\cdot\begin{bmatrix}\frac{3}{2}\\ 1\\ 1\end{bmatrix} dydx \\
&= -\int_0^2\int_0^{3-\frac{3}{2}x} 8xdydx \\
&= \int_0^2 -24x+12x^2dx \\
&= -12x^2+4x^3\Big|_0^2 \\
&= -16.
\end{aligned}$$

Example 99. Let $\vec{F} = y\vec{i} - x\vec{j} + zx^3y^2\vec{k}$, and let S be the portion of the cone $z = \sqrt{z^2+y^2}$ where $1 \le z \le 2$, oriented "inward/upward". Compute $\iint_S (\text{curl}\vec{F})\cdot d\vec{S}$.

Let's parametrize the cone with

$$\vec{X}(r,\theta) = (r\cos(\theta), r\sin(\theta), r), \quad 1\le r\le 2,\ 0\le\theta\le 2\pi.$$

For this parametrization, we have

$$\vec{X}_r(r,\theta) = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\\ 1\end{bmatrix} \quad \text{and} \quad \vec{X}_\theta(r,\theta) = \begin{bmatrix}-r\sin(\theta)\\ r\cos(\theta)\\ 0\end{bmatrix}, \quad \text{so that} \quad N_{\vec{X}}(r,\theta) = \begin{bmatrix}-r\cos(\theta)\\ -r\sin(\theta)\\ r\end{bmatrix}.$$

This normal vector points in the correct direction, so $\vec{X}$ is an orientation-preserving parametrization of S and there is no need to correct.

After computing $\text{curl}\vec{F} = 2zx^3y\vec{i} - 3zx^2y^2\vec{j} - 2\vec{k}$, we have

$$\begin{aligned}
\iint_S (\text{curl}\vec{F})\cdot d\vec{S} &= \int_1^2\int_0^{2\pi} \begin{bmatrix}2r^5\cos^3(\theta)\sin(\theta)\\ -3r^5\cos^2(\theta)\sin^2(\theta)\\ -2\end{bmatrix}\cdot N_{\vec{X}}(r,\theta)d\theta dr \\
&= \int_1^2\int_0^{2\pi}\Big(3r^6\cos^2(\theta)\sin^3(\theta) - 2r^6\cos^4(\theta)\sin(\theta) - 2r\Big)d\theta dr \\
&= \int_1^2 \Big[3r^6\Big(\frac{\cos^3(\theta)}{3} - \frac{\cos^5(\theta)}{5}\Big) + \frac{2}{5}r^6\cos^5(\theta) - 2r\theta\Big|_0^{2\pi} dr \\
&= \int_1^2 -4\pi r dr = -6\pi.
\end{aligned}$$

Lecture 26: Stokes' Theorem

Learning Objectives:

- Apply Stokes' Theorem to relate vector surface integrals with vector line integrals.

Now that we have a notion of vector surface integral, the final two generalizations of the Fundamental Theorem of Calculus that we will discuss are in reach. The first, Stokes' Theorem, is essentially for surfaces in $\mathbb{R}^3$ what Green's Theorem was for regions in $\mathbb{R}^2$. Because the (geometric) boundary of a surface consists of curves, we expect Stokes' Theorem to relate vector surface integrals with vector line integrals. In terms of differential forms, Stokes' Theorem should be

$$\iint_S d\omega = \int_{\partial S} \omega$$

where $\omega = P dx + Q dy + R dz$ is a 1-form. But we've seen that if $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is the vector field associated with the 1-form ω, then $\text{curl}\vec{F}$ is the vector field associated with the 2-form

$$d\omega = (R_y - Q_z)\, dy \wedge dz + (P_z - R_x)\, dz \wedge dx + (Q_x - P_y)\, dx \wedge dy.$$

Therefore, in terms of vector fields, Stokes' Theorem should have the form

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}.$$

Of course, it still isn't obvious what should be the orientation of ∂S. As with Green's Theorem, though, the correct orientation of ∂S falls out in the proof.

Theorem 16 (Stokes' Theorem)**.** Let $S \subset \mathbb{R}^3$ be an oriented piecewise-smooth surface such that ∂S consists of a finite union of closed piecewise-smooth curves $C_1, \ldots, C_k$, where each C_j is oriented so that, while traveling along C_j, the surface S is "on the left" when viewed from "above" (where "up" at each point is the direction specified by the orientation of S). If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a C^1 vector field on an open set $U \subseteq \mathbb{R}^3$ with $S \subset U$, then

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s} = \oint_{C_1} \vec{F} \cdot d\vec{s} + \cdots + \oint_{C_k} \vec{F} \cdot d\vec{s}.$$

In terms of differential forms, this can be expressed as

$$\iint_S (R_y - Q_z)\, dy \wedge dz + (P_z - R_x)\, dz \wedge dx + (Q_x - P_y)\, dx \wedge dy = \int_{\partial S} P dx + Q dy + R dz.$$

Sketch of Proof of Stokes' Theorem. The proof of Stokes' Theorem very similar to that of Green's Theorem, but is slightly easier because we can actually *use* Green's Theorem to prove it.

For the first step (and this involves some analysis), we break S into a finite number of smooth surfaces such that each can be viewed as a graph of one variable as a function of the other two. (E.g. a piece of S has the form $(x, g(x,z), z)$ for (x,z) in some region of the xz-plane.) Each piece will require a different representation, and we can even ensure that the boundary of each piece consists of a single piecewise-smooth closed curve.

Next, we prove Stoke's Theorem for each piece of surface constructed in the first step. You will actually do prove this step on your homework! Ultimately, the proof of this involves parametrizing the vector surface and vector line integrals, and noting that the results are related to each other via Green's Theorem.

The final step is to "piece the results back together". The sum of the vector surface integrals of $\text{curl}\vec{F}$ over the pieces of S will give $\iint_S \text{curl}\vec{F}\cdot d\vec{S}$, and one can show (using a "splitting and cancellation" argument of the type used in Green's Theorem) that the sum of the vector line integrals of $\vec{F}$ over the boundaries of the pieces of S will give $\oint_{\partial S} \vec{F}\cdot d\vec{s}$. □

Remark 54. The physical interpretation of Stokes' Theorem is very similar to the physical interpretation of Green's Theorem. That is, the integral on the right-hand-side of

$$\iint_S \text{curl}\vec{F}\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{s}$$

measures the circulation of $\vec{F}$ around ∂S in the direction that ∂S is oriented (i.e. so that S is "on the left" when viewed from "above", where "above" is the direction that the normal to S points).

On the other hand, $\text{curl}\vec{F}\cdot\vec{n}$ measures how much of $\text{curl}\vec{F}$ points in the direction of $\vec{n}$. Because the magnitude of $\text{curl}\vec{F}$ measures "twisting at a point" in the plane perpendicular to $\text{curl}\vec{F}$, $\text{curl}\vec{F}\cdot\vec{n}$ measures the "twisting at a point" on S *that is tangent to* S. Therefore Stokes' Theorem is a way to compare the total net twisting of $\vec{F}$ on S with the circulation of $\vec{F}$ around ∂S.

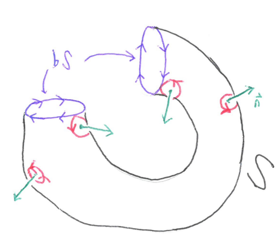
Remark 55. Our intuition above for what $\text{curl}\vec{F}$ means can actually be *justified* using Stokes' Theorem. Indeed, if $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a C^1 vector field on $\mathbb{R}^3$, and $\text{curl}\vec{F}(x_0, y_0, z_0) \neq \vec{0}$, then let S_r denote the surface consisting of points (x,y,z) on the plane through (x_0,y_0,z_0) that is normal to $\text{curl}\vec{F}(x_0,y_0,z_0)$, such that $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq r^2$. Orient S_r so that the unit normal vectors $\vec{n}$ point in the same direction as $\text{curl}\vec{F}(x_0,y_0,z_0)$. Then one can show that

$$\|\text{curl}\vec{F}(x_0,y_0,z_0)\| = \lim_{r\to 0+} \frac{1}{\pi r^2} \oint_{\partial S_r} \vec{F}\cdot d\vec{s},$$

where the circle ∂S_r is oriented so that, when viewed from "above" (i.e. from the direction of $\text{curl}\vec{F}(x_0,y_0,z_0)$), S_r is "on the left".

Remark 56. Here is an equivalent way to characterize the orientation of the C_j required by Stokes' Theorem.

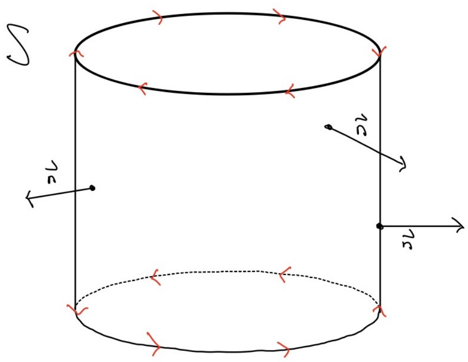
If one picks a point $\vec{p} \in S$ and draws the unit normal vector $\vec{n}$ to S at $\vec{p}$, then one can think of a small disc at the base of $\vec{n}$ (i.e. in S) rotating counterclockwise when viewed from the positive $\vec{n}$ direction.

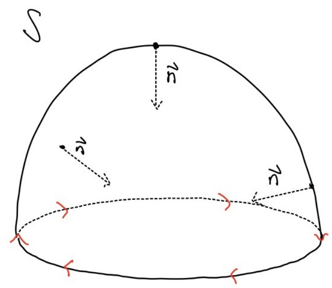


This little spinning disc will determine the orientation of each piece of ∂S; we merely choose a point $\vec{p}$ near the boundary, and give that piece of boundary the orientation that it receives from the spinning disc, as shown in the above picture.

Remark 57. The orientation of ∂S that described in the statement of Stokes' Theorem is sometimes referred to as the orientation of ∂S that is **induced** by the orientation of S.

Example 100. For each of the following two surfaces S, describe the orientation of each piece of ∂S which is compatible with the orientation of S.



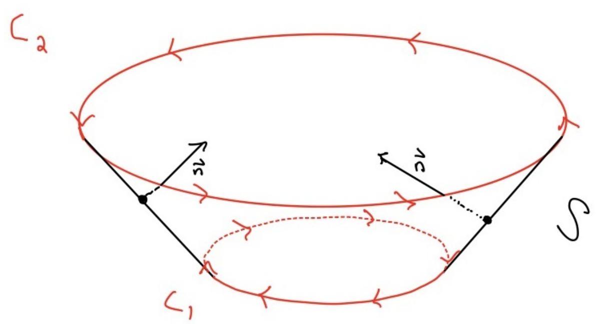


In the first picture, the cylinder with outward-pointing normal vector, the boundary is composed of two circles. In order to be oriented compatibly with S, the top circle must be oriented clockwise (when viewed from above). The bottom circle, however, must be oriented counterclockwise (when viewed from above).

In the second picture, the portion of the sphere with inward-pointing normal vector, we see that the boundary circle must be parametrized clockwise (when viewed from above) in order to have the orientation compatible with that of S.

Example 101. Let $\vec{F} = y\vec{i} - x\vec{j} + zx^3y^2\vec{k}$, and let S be the portion of the cone $z = \sqrt{x^2+y^2}$ where $1 \le z \le 2$, oriented "inward/upward". Compute $\iint\limits_S (\text{curl}\vec{F}) \cdot d\vec{S}$ using Stokes' Theorem.

(Note that we've already solved this problem directly without Stokes' Theorem, and while doable the problem became quite messy near the end.) The boundary of this surface consists of the circles C_2: $x^2+y^2=4,\ z=2$ and C_1: $x^2+y^2=1,\ z=1$. Since the cone has upward-pointing normal vector, we see that we should orient C_2 counterclockwise and orient C_1 clockwise (both when viewed from above on the positive z-axis).



We parametrize C_2 with

$$\vec{r}_2(\theta) = (2\cos(\theta), 2\sin(\theta), 2),\ \ 0 \le \theta \le 2\pi$$

and parametrize $-C_1$ with

$$\vec{r}_1(\theta) = (\cos(\theta), \sin(\theta), 1),\ \ 0 \le \theta \le 2\pi,$$

and apply Stokes' Theorem to get

$$\begin{aligned}
\iint\limits_S (\text{curl}\vec{F}) \cdot d\vec{S} &= \oint_{\partial S} \vec{F} \cdot d\vec{S} \\
&= \oint_{C_2} \vec{F} \cdot d\vec{s} + \oint_{C_1} \vec{F} \cdot d\vec{s} \\
&= \oint_{C_2} \vec{F} \cdot d\vec{s} - \oint_{-C_1} \vec{F} \cdot d\vec{s} \\
&= \int_0^{2\pi} \begin{bmatrix} 2\sin(\theta) \\ -2\cos(\theta) \\ \text{whatever} \end{bmatrix} \cdot \begin{bmatrix} -2\sin(\theta) \\ 2\cos(\theta) \\ 0 \end{bmatrix} d\theta - \int_0^{2\pi} \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ \text{whatever} \end{bmatrix} \cdot \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} d\theta \\
&= \int_0^{2\pi} -4 d\theta + \int_0^{2\pi} 1 d\theta \\
&= -6\pi.
\end{aligned}$$

Much easier!

Lecture 27: More Stokes' Theorem

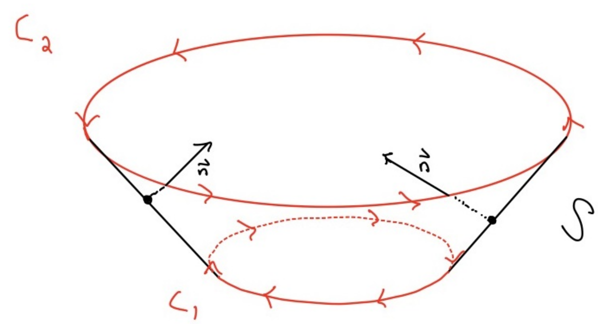
Learning Objectives:

- Apply Stokes' Theorem to relate vector surface integrals with vector line integrals.
- Investigate deeper application of Stokes' Theorem, such as replacing one surface integral with another or seeing what happens in the case of closed surfaces.

We start with an example that we started, but did not complete, last time.

Example 102. Let $\vec{F} = y\vec{i} - x\vec{j} + zx^3y^2\vec{k}$, and let S be the portion of the cone $z = \sqrt{x^2+y^2}$ where $1 \le z \le 2$, oriented "inward/upward". Compute $\iint_S (\text{curl}\vec{F}) \cdot d\vec{S}$ using Stokes' Theorem.

(Note that we've already solved this problem directly without Stokes' Theorem, and while doable the problem became quite messy near the end.) The boundary of this surface consists of the circles C_2: $x^2+y^2=4,\ z=2$ and C_1: $x^2+y^2=1,\ z=1$. Since the cone has upward-pointing normal vector, we see that we should orient C_2 counterclockwise and orient C_1 clockwise (both when viewed from above on the positive z-axis).



We parametrize C_2 with

$$\vec{r}_2(\theta) = (2\cos(\theta), 2\sin(\theta), 2),\ 0 \le \theta \le 2\pi$$

and parametrize $-C_1$ with

$$\vec{r}_1(\theta) = (\cos(\theta), \sin(\theta), 1),\ 0 \le \theta \le 2\pi,$$

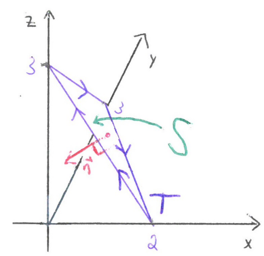
and apply Stokes' Theorem to get

$$\begin{aligned}
\iint_S (\text{curl}\vec{F}) \cdot d\vec{S} &= \oint_{\partial S} \vec{F} \cdot d\vec{S} = \oint_{C_2} \vec{F} \cdot d\vec{s} + \oint_{C_1} \vec{F} \cdot d\vec{s} \\
&= \oint_{C_2} \vec{F} \cdot d\vec{s} - \oint_{-C_1} \vec{F} \cdot d\vec{s} \\
&= \int_0^{2\pi} \begin{bmatrix} 2\sin(\theta) \\ -2\cos(\theta) \\ \text{whatever} \end{bmatrix} \cdot \begin{bmatrix} -2\sin(\theta) \\ 2\cos(\theta) \\ 0 \end{bmatrix} d\theta - \int_0^{2\pi} \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ \text{whatever} \end{bmatrix} \cdot \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} d\theta \\
&= \int_0^{2\pi} -4 d\theta + \int_0^{2\pi} 1 d\theta \\
&= -6\pi.
\end{aligned}$$

Much easier!

Example 103. Let T be the triangle in $\mathbb{R}^3$ with vertices $(2,0,0)$, $(0,3,0)$, $(0,0,3)$, oriented clockwise when viewed from above. Compute $\oint_T \vec{F}\cdot d\vec{s}$, where $\vec{F}(x,y,z) = (4yz, 2+y^2e^y, 4z\sin(x))$.

We sketch T below. As we see, T is the boundary of the portion S of the plane $3x+2y+2z=6$ in the first octant from Example 97.



We would like to apply Stokes' Theorem to the line integral in order to get $\iint_S(\text{curl}\vec{F})\cdot d\vec{S}$. Before we can do this, though, we need to first figure out what will be the appropriate orientation for S, given the orientation of bS. By inspecting the orientation of T, we see that S *must be given the downward orientation.* Using the parametrization from last time,

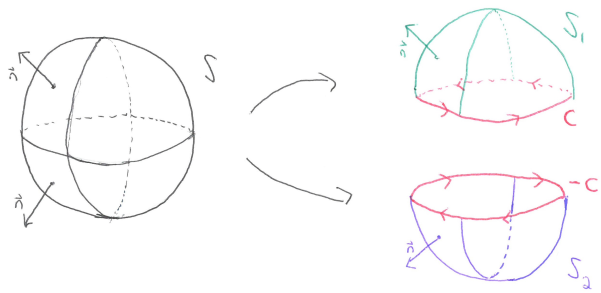
$$\vec{X}(x,y) = (x, y, 3-y-\frac{3}{2}x),\ 0\le x\le 2,\ 0\le y\le 3-\frac{3}{2}x, \quad N_{\vec{X}}(x,y) = \begin{bmatrix}\frac{3}{2}\\1\\1\end{bmatrix},$$

(which is orientation-reversing since its positive $\vec{k}$-component implies that it points upward, which is the opposite of what we want) we have

$$\begin{aligned}
\oint_T \vec{F}\cdot d\vec{s} &= \oint_{\partial S}\vec{F}\cdot d\vec{s}\\
&= \iint_S(\text{curl}\vec{F})\cdot d\vec{S} = -\iint_{-S}(\text{curl}\vec{F})\cdot d\vec{S}\\
&= -\int_0^2\int_0^{3-\frac{3}{2}x}\text{curl}\vec{F}\Big(x,y,3-y-\frac{3}{2}x\Big)\cdot N_{\vec{X}}(x,y)dydx\\
&= -\int_0^2\int_0^{3-\frac{3}{2}x}\begin{bmatrix}0\\4y-4(3-y-\frac{3}{2}x)\cos(x)\\-4(3-y-\frac{3}{2}x)\end{bmatrix}\cdot\begin{bmatrix}\frac{3}{2}\\1\\1\end{bmatrix}dydx\\
&= \int_0^2\int_0^{3-\frac{3}{2}x}-4y+4\Big(3-y-\frac{3}{2}x\Big)\cos(x)+4\Big(3-y-\frac{3}{2}x\Big)dydx\\
&= \int_0^2\int_0^{3-\frac{3}{2}x}-8y+12\cos(x)-4y\cos(x)-6x\cos(x)+12-6xdydx\\
&= 18-9\sin(2).
\end{aligned}$$

Example 104. Let's compute $\iint_S(\text{curl}\vec{F})\cdot d\vec{S}$, where S is the unit sphere $x^2+y^2+z^2=1$ (with the outward orientation) and $\vec{F} = \cos(e^{xyz})\vec{i}+\sin(xyz+e^z)\vec{j}+(x^2+1)^{zy}\vec{k}$.

We would like to apply Stokes' Theorem here, but something seems a little fishy: the sphere S has no (geometric) boundary! Something very interesting will happen here, but for now let's just split the sphere into two pieces: the upper hemisphere S_1 and the lower hemisphere S_2.



The boundary of each is the unit circle $x^2+y^2=1$ in the xy-plane. However, the orientation of the circle which is compatible with the orientation of S_1 (counterclockwise when viewed from *above*) is different than that which is compatible with the orientation of S_2 (counterclockwise when viewed from *below*).

Therefore, if C is the circle $x^2+y^2=1,\ z=0$ oriented counterclockwise when viewed from above, we have

$$\begin{aligned}\iint_S (\text{curl}\vec{F})\cdot d\vec{S} &= \iint_{S_1} (\text{curl}\vec{F})\cdot d\vec{S} + \iint_{S_2} (\text{curl}\vec{F})\cdot d\vec{S} \\ &= \oint_C \vec{F}\cdot d\vec{s} + \oint_{-C} \vec{F}\cdot d\vec{s} \\ &= 0.\end{aligned}$$

Unlike many of the integrals where we get 0, there is no 'symmetry' going on in the previous example. The important thing here was that S is a **closed** surface, meaning that the geometric boundary of S is empty. That is, $\partial S=\emptyset$. By abstracting the argument used in the previous example one can show that

Theorem 17. If $S\subset\mathbb{R}^3$ is an oriented, closed, piecewise-smooth surface and if $\vec{F}$ is a C^1 vector field defined on an open set $U\subseteq\mathbb{R}^3$ containing S, then

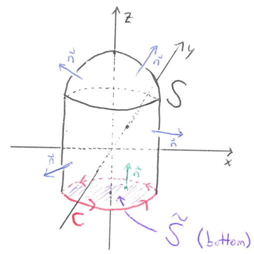
$$\iint_S (\text{curl}\vec{F})\cdot d\vec{S}=0.$$

Remark 58. Note how closely related this is to the theorem about conservative vector fields, which said that $\int_C \nabla f\cdot d\vec{s}=0$ as long as C is a closed curve. Indeed, this is just the '2-dimensional version' of the conservative vector field result!

After learning Gauss's theorem, we will discuss this theorem again (with a new interpretation). For now, let's see an application.

Example 105. Let S be the (outward oriented) surface formed by the portion of the cylinder $x^2+y^2=1,\ |z|\le 1$, together with the upper half of the sphere $x^2+y^2+(z-1)^2=1$. Compute $\iint_S (\text{curl}\vec{F})\cdot d\vec{S}$, where $\vec{F}=(x^3-y^3)e^{\sin(\pi z)}\vec{i}+(x^3+\sin(z^2-1))\vec{j}+e^{xy^2}z\cos(x^2+yz)\vec{k}$.

We sketch the surface S below:



The boundary of S is the circle $x^2+y^2=1,\ z=-1$. Let C be this circle, oriented in the counterclockwise direction when viewed from above. We will illustrate two different (but very similar) ways to use Stokes' theorem to replace the surface S with a different surface that has the same boundary.

To do this, let $\tilde{S}$ denote the disc $x^2+y^2 \leq 1$ in the plane $z=-1$, oriented upward. Then C is also the boundary of $\tilde{S}$, and the orientations of C and $\tilde{S}$ are compatible. Hence, by applying Stokes' Theorem twice,

$$\iint_S (\text{curl}\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{s} = \iint_{\tilde{S}} (\text{curl}\vec{F}) \cdot d\vec{S}.$$

Alternatively, we could 'cap off' the surface S with $-\tilde{S}$ (i.e. the surface $\tilde{S}$ but with the opposite orientation as $\tilde{S}$. That is, $S \cup (-\tilde{S})$ is a closed, oriented (outward) piecewise-smooth surface. Therefore, by the previous theorem,

$$\begin{aligned}\iint_S (\text{curl}\vec{F}) \cdot d\vec{S} &= \iint_S (\text{curl}\vec{F}) \cdot d\vec{S} + \iint_{-\tilde{S}} (\text{curl}\vec{F}) \cdot d\vec{S} + \iint_{\tilde{S}} (\text{curl}\vec{F}) \cdot d\vec{S} \\ &= 0 + \iint_{\tilde{S}} (\text{curl}\vec{F}) \cdot d\vec{S} = \iint_{\tilde{S}} (\text{curl}\vec{F}) \cdot d\vec{S}.\end{aligned}$$

At any rate, we need only compute $\iint_{\tilde{S}} (\text{curl}\vec{F}) \cdot d\vec{S}$. To do this, we use the parametrization

$$\vec{X}(r,\theta) = (r\cos(\theta), r\sin(\theta), -1),\ 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi.$$

For this parametrization, we have $\vec{N}(r,\theta) = r\vec{k}$ (which agrees with the orientation of $\tilde{S}$), so that

$$\begin{aligned}\iint_S (\text{curl}\vec{F}) \cdot d\vec{S} &= \iint_{\tilde{S}} (\text{curl}\vec{F}) \cdot d\vec{S} \\ &= \iint_{\tilde{S}} \begin{bmatrix} \textit{Garbage} \\ \textit{More Garbage} \\ 3x^2+3y^2e^{\sin(\pi z)} \end{bmatrix} \cdot d\vec{S} \\ &= \int_0^1 \int_0^{2\pi} \begin{bmatrix} \textit{Garbage} \\ \textit{More Garbage} \\ 3r^2 \end{bmatrix} \cdot (r\vec{k}) d\theta dr \\ &= \int_0^1 \int_0^{2\pi} 3r^3 d\theta dr \\ &= \frac{3\pi}{2}.\end{aligned}$$

The first method illustrated in the above example can be generalized as follows.

Theorem 18. If $S_1, S_2 \subset \mathbb{R}^3$ are oriented, piecewise-smooth surfaces with common boundary C, where the orientation of C is (simultaneously) the orientation induced by the orientations both S_1 and S_2, then

$$\iint_{S_1} (\text{curl}\vec{F}) \cdot d\vec{S} = \iint_{S_2} (\text{curl}\vec{F}) \cdot d\vec{S}$$

for any C^1 vector field $\vec{F}$ defined on (an open set containing) both S_1 and S_2.

Both of these theorems can be quite useful! Be on the lookout for applications of them (i.e. whenever you have to integrate over some nasty surface).

Lecture 28: Gauss's Theorem

Learning Objectives:

- Apply Gauss's Theorem to relate vector surface integrals with vector line integrals.
- Investigate deeper applications of Gauss's Theorem, such as replacing one surface integral with another.

We now discuss our very last result in the course, which is also a generalization of the Fundamental Theorem of Calculus. Our final result will be for a region $E \subset \mathbb{R}^3$ such that ∂E consists of closed piecewise-smooth surfaces, and will have the form

$$\iiint_E d\omega = \iint_{\partial E} \omega,$$

where $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$ is a C^1 2-form. But we have seen that

$$d\omega = (P_x + Q_y + R_z)\,dx \wedge dy \wedge dz.$$

In other words, if $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is the C^1 vector field corresponding to ω, then the function $\text{div}\vec{F}$ corresponds to $d\omega$. Our theorem, due to Gauss, is stated below.

Theorem 19 (Gauss's Theorem)**.** Let $E \subseteq \mathbb{R}^3$ be a region whose boundary ∂E consists of a finite union of closed piecewise-smooth surfaces $S_1, \ldots, S_k$, where each S_j is oriented with normal vectors that point "out of" E. If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a C^1 vector field on an open set $U \subseteq \mathbb{R}^3$ with $E \subseteq U$, then

$$\iiint_E \text{div}\vec{F} dV(x,y,z) = \oiint_{\partial E} \vec{F} \cdot d\vec{S} \stackrel{def}{=} \oiint_{S_1} \vec{F} \cdot d\vec{S} + \cdots + \oiint_{S_k} \vec{F} \cdot d\vec{S}.$$

In terms of differential forms this can be expressed as

$$\iiint_E (P_x + Q_y + R_z)\,dx \wedge dy \wedge dz = \iint_{\partial E} P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$$

Here, $\oiint$ denotes that the integral is over a (collection of) closed surface(s), and has no additional meaning.

Remark 59. One important thing to note is that the orientation 'outward' is determined relative to the region E. For example, if E is the region in $\mathbb{R}^3$ defined by $1 \le x^2 + y^2 + z^2 \le 4$, then ∂E consists of two spheres, $S_1 : \ x^2 + y^2 + z^2 = 1$ and $S_2 : \ x^2 + y^2 + z^2 = 4$. If we wanted to apply Gauss's Theorem to a triple integral over E, then we would need to orient each of these spheres so that their normal vectors point away from E. That is, S_2 is given the normal vector that points away from $(0,0,0)$, but S_1 must be given the normal vector that points *towards* $(0,0,0)$.

As always, if the orientation of S is incorrect (i.e. if the normal vector points *into* the region E), or if we accidentally produce an orientation-reversing parametrization of S, then we can change the orientation at the cost of a minus sign. We'll see an example of this shortly.

Remark 60. The physical intuition for Gauss's Theorem is as follows. If $\vec{F}$ describes the flow of some fluid in $\mathbb{R}^3$, then the integral over ∂E measures the net amount of fluid flowing outward through the boundary of E (with negative values representing flow *into* E). On the other hand, $\text{div}\vec{F}(\vec{p})$ measures the amount of expansion (or contraction, if negative) of the fluid at $\vec{p}$, so the triple integral over E measures the net amount of expansion (and contraction, which contributes negatively to the value of the integral) of the fluid inside of E. In other words, Gauss's theorem relates the net expansion of the fluid within E to the amount of fluid that flows outward through ∂E. For this reason, Gauss's Theorem is also called the **Divergence Theorem**.

Remark 61. Just as Green's and Stokes' Theorems allowed us to justify our understanding of the physical meaning of curl, Gauss's Theorem justifies our understanding of divergence as a measure of expansion or contraction of a fluid. Indeed, if $\vec{F}$ is a C^1 vector field on $\mathbb{R}^3$, then $\text{div}\vec{F}(\vec{x})$ is a continuous function on $\mathbb{R}^3$, and therefore by Exercise 4 in Homework 4 we have, for each $\vec{x}_0 \in \mathbb{R}^3$ we have

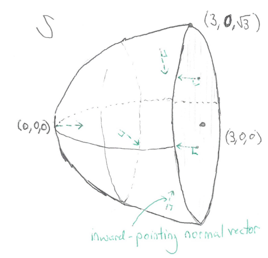
$$\text{div}\vec{F}(\vec{x}_0) = \lim_{r\to 0+} \frac{1}{\text{Vol}_3(B_r(\vec{x}_0))} \iiint_{B_r(\vec{x}_0)} \text{div}\vec{F}(\vec{x})\, dV_3(\vec{x}) = \lim_{r\to 0+} \frac{1}{\text{Vol}_3(B_r(\vec{x}_0))} \iint_{\partial B_r(\vec{x}_0)} \vec{F}\cdot d\vec{S}$$

by Gauss's Theorem, where the sphere $\partial B_r(\vec{x}_0)$ is oriented with "outward-pointing" normals.

Therefore we have that $\text{div}\vec{F}(\vec{x}_0)$ is a measurement of "net infinitesimal flux" out of a sphere centered at $\vec{x}_0$. When $\text{div}\vec{F}(\vec{x}_0) > 0$ then more fluid is flowing out of this sphere than into it, so it must be that the fluid described by $\vec{F}$ is "expanding" at $\vec{x}_0$. If $\text{div}\vec{F}(\vec{x}_0) < 0$, then this fluid must be "contracting" at $\vec{x}_0$.

Example 106. Let S be the piecewise-smooth oriented surface consisting of the portion of the paraboloid $x = y^2 + z^2$ between the yz-plane and the plane $x = 3$, with 'inward pointing' orientation, together with the portion of the plane $x = 3$ with orientation $-\vec{i}$ (i.e. orientation pointing towards the yz-plane). Compute $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = (2xz^2 + x^3, 3xz, 2zy^2)$.

We sketch the surface S below:



Note that S is the boundary of the region E bounded between the paraboloid $x = y^2 + z^2$ and the plane $x = 3$. We would therefore like to apply Gauss's Theorem. However, the orientation of S points *towards* E, not away from it! The fix is easy: we can replace S with $-S$ (i.e. S with the opposite orientation) at the expense of a negative sign. Therefore, we apply Gauss's Theorem to get

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_{-S} \vec{F}\cdot d\vec{S} = -\oiint_{\partial E} \vec{F}\cdot d\vec{S} = -\iiint_E \text{div}\vec{F} dV = -\iiint_E (3x^2 + 2y^2 + 2z^2) dV(x,y,z).$$

To evaluate this last integral, we'll use an adaptation of cylindrical coordinates:

$$(x,y,z) = (x, r\cos(\theta), r\sin(\theta)),\ 0 \le r \le \sqrt{3},\ 0 \le \theta \le 2\pi,\ r^2 \le x \le 3.$$

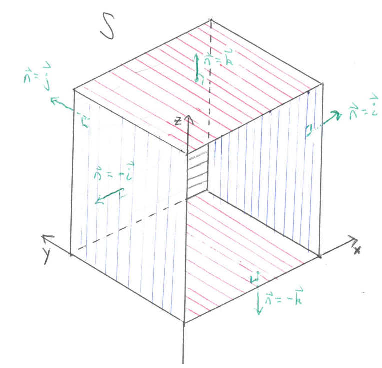
The Jacobian of this change of variables is

$$\left|\frac{\partial(x,y,z)}{\partial(x,r,\theta)}\right| = \left|\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -r\sin(\theta) \\ 0 & \sin(\theta) & r\cos(\theta) \end{bmatrix}\right| = r,$$

so that

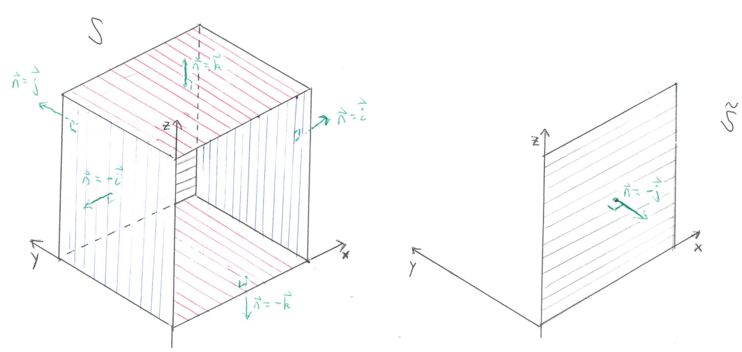
$$\begin{aligned} \iint\limits_S \vec{F}\cdot d\vec{S} &= -\iiint\limits_E (3x^2+2y^2+2z^2)dV(x,y,z) \\ &= -\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r^2}^{3}(3x^2+2r^2)r\,dx\,dr\,d\theta \\ &= -\int_0^{2\pi}\int_0^{\sqrt{3}}(-r^7-2r^5+6r^3+27r)\,dr\,d\theta \\ &= -\int_0^{2\pi}\Big(-\frac{81}{8}-9+\frac{27}{2}+\frac{81}{2}\Big)d\theta \\ &= -\frac{279\pi}{4}. \end{aligned}$$

Example 107. Let S be the boundary of the box $[0,1]\times[0,1]\times[0,1]$, except for the face on the xz-plane, and give S the 'outward' orientation (i.e. 'away from $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$'). Compute $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = (2xy, x^2z^2, 3z + x\sin(y))$.



Because S is not closed, we cannot directly apply Gauss's Theorem. However, the prospect of computing five different surface integrals seems particularly tedius (even though they are not really that bad)! We will therefore reintroduce our trick of 'closing off' the surface S so that we *can* apply Gauss's Theorem.

To this end, let $\tilde{S}$ be the square $(x,0,z)$, $0\le x\le 1$, $0\le z\le 1$, oriented so that its normal vector is $-\vec{j}$.



Then $S \cup \tilde{S}$ <u>is</u> a closed surface, and is the boundary of the unit cube $E = [0,1] \times [0,1] \times [0,1]$ with outward pointing normal vector. We can therefore apply Gauss's theorem to this closed surface. That is, we have

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \underbrace{\iint_S \vec{F} \cdot d\vec{S} + \iint_{\tilde{S}} \vec{F} \cdot d\vec{S}}_{\displaystyle \oiint_{\partial E} \vec{F} \cdot d\vec{S}} - \iint_{\tilde{S}} \vec{F} \cdot d\vec{S} \\ &= \iiint_E \mathrm{div}\vec{F} dV - \iint_{\tilde{S}} \vec{F} \cdot d\vec{S}.\end{aligned}$$

We evaluate the triple integral using our old friend Fubini:

$$\iiint_E \mathrm{div}\vec{F} dV = \iiint_E (2y + 0 + 3) dV(x,y,z) = \int_0^1 \int_0^1 \int_0^1 (2y+3) dx dz dy = \int_0^1 (2y+3) dy = 4.$$

For the integral over $\tilde{S}$, we use the fact that $\vec{n} = -\vec{j}$ to write

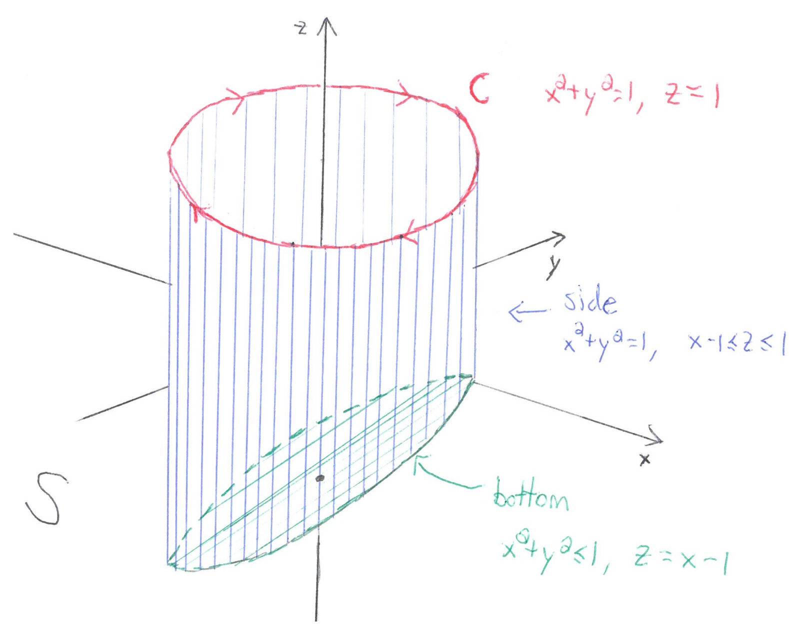
$$\iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iint_{\tilde{S}} \vec{F} \cdot (-\vec{j}) dS = \iint_{\tilde{S}} -x^2 z^2 dS = \int_0^1 \int_0^1 -x^2 z^2 dx dz = -\frac{1}{9}.$$

In summary,

$$\iint_S \vec{F} \cdot d\vec{S} = 4 + \frac{1}{9} = \frac{37}{9}.$$

Example 108. Let S be the (outward-oriented) surface formed by the portion of the cylinder $x^2+y^2 = 1$ below the plane $z = 1$ and above the plane $z = x - 1$, together with the bottom 'shorn disc' $x^2 + y^2 \leq 1,\ z = x - 1$. Let $\vec{G} = (2y, x, 14z^2 - xy)$ and $\vec{F} = (0, x^2 e^{z-x} - \frac{2}{3} x y^3, x + x^2 y^2)$. Compute

$$\iint_S (\mathrm{curl}\vec{G} + \vec{F}) \cdot d\vec{S}.$$



Since $\text{div}\vec{F} \neq 0$, we cannot hope to write $\vec{F} = \text{curl}\tilde{\vec{F}}$ for some vector field $\tilde{\vec{F}}$ (and thereby apply Stokes' Theorem to the entire integral). However, we can split the integral into two pieces

$$\iint_S (\text{curl}\vec{G} + \vec{F}) \cdot d\vec{S} = \iint_S (\text{curl}\vec{G}) \cdot d\vec{S} + \iint_S \vec{F} \cdot d\vec{S},$$

and deal with each piece separately.

For the first piece, let's apply Stokes' Theorem. The boundary of S is the circle $x^2 + y^2 = 1$ in the plane $z = 1$. Let C be this circle, oriented clockwise when viewed from above. Then the orientation of C is compatible with that of S, so by parametrizing C with $(x, y, z) = (\cos(t), -\sin(t), 1)$, $0 \leq t \leq 2\pi$, we have

$$\begin{aligned}
\iint_S (\text{curl}\vec{G}) \cdot d\vec{S} &= \oint_C \vec{G} \cdot d\vec{s} \\
&= \int_0^{2\pi} \begin{bmatrix} -2\sin(t) \\ \cos(t) \\ 14(1)^2 + \sin(t)\cos(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ -\cos(t) \\ 0 \end{bmatrix} dt \\
&= \int_0^{2\pi} 2\sin^2(t) - \cos^2(t) dt \\
&= \int_0^{2\pi} \frac{1}{2} - \frac{3}{2}\cos(2t) dt \\
&= \pi.
\end{aligned}$$

For the second piece we'd like to apply Gauss's Theorem. However, because S is not closed, we need to 'cap it off' by adding in the missing piece. Let $\tilde{S}$ be the portion of the plane $z = 1$ which is inside of the cylinder $x^2 + y^2 = 1$. In order for the orientation of $\tilde{S}$ to be consistent with that of S, we must give $\tilde{S}$ the upward-pointing orientation. Therefore, if E is the region of $\mathbb{R}^3$ bounded by S and $\tilde{S}$, then the boundary $S \cup \tilde{S}$ of E is oriented outward, allowing us to apply Gauss's Theorem. That is,

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \iint_{S \cup \tilde{S}} \vec{F} \cdot d\vec{S} - \iint_{\tilde{S}} \vec{F} \cdot d\vec{S} \\
&= \iiint_W \text{div}\vec{F} dV - \iint_{\tilde{S}} \vec{F} \cdot d\vec{S} \\
&= \iiint_W -2xy^2 dV(x, y, z) - \iint_{\tilde{S}} \vec{F} \cdot d\vec{S}.
\end{aligned}$$

To evaluate the triple integral, we use cylindrical coordinates with z as the (inner)-variable:

$$\begin{aligned}
\iiint_W -2xy^2 dV(x,y,z) &= \int_0^1 \int_0^{2\pi} \int_{r\cos(\theta)-1}^1 -2r^4\cos(\theta)\sin^2(\theta) dz d\theta dr \\
&= \int_0^1 \int_0^{2\pi} (-4r^4\cos(\theta)\sin^2(\theta) + 2r^5\cos^2(\theta)\sin^2(\theta)) d\theta dr \\
&= \int_0^1 \int_0^{2\pi} (-4r^4\cos(\theta)\sin^2(\theta) + \frac{1}{2}r^5\sin^2(2\theta)) d\theta dr \\
&= \int_0^1 \int_0^{2\pi} (-4r^4\cos(\theta)\sin^2(\theta) + \frac{1}{4}r^5 - \frac{1}{4}r^5\cos(4\theta)) d\theta dr \\
&= \int_0^1 \left[-\frac{4}{3}r^4\sin^3(\theta) + \frac{1}{4}r^5\theta - \frac{1}{16}r^5\sin(4\theta) \right|_0^{2\pi} dr \\
&= \int_0^1 \frac{\pi}{2} r^5 dr \\
&= \frac{\pi}{12}.
\end{aligned}$$

For the surface integral, we use the fact that $\vec{n} = \vec{k}$ at each point of $\tilde{S}$ to first write

$$\iint_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iint_{\tilde{S}} (x + x^2y^2) dS = \iint_{\tilde{S}} x^2y^2 dS,$$

where we used the fact that x is an odd function in x and $\tilde{S}$ is symmetric with respect to the yz-plane. To compute this last integral, we just use polar coordinates in the plane:

$$\begin{aligned}
\iint_{\tilde{S}} x^2y^2 dS &= \int_0^{2\pi} \int_0^1 r^5 \sin^2(\theta)\cos^2(\theta) dr d\theta \\
&= \frac{1}{6}\int_0^{2\pi} \sin^2(\theta)\cos^2(\theta) d\theta \\
&= \frac{1}{48}\int_0^{2\pi} 1 - \cos(4\theta) d\theta \\
&= \frac{\pi}{24}.
\end{aligned}$$

Putting all of these together,

$$\iint_S (\text{curl}\vec{G} + \vec{F}) \cdot d\vec{S} = \pi + \frac{\pi}{12} - \frac{\pi}{24} = \frac{25\pi}{24}.$$