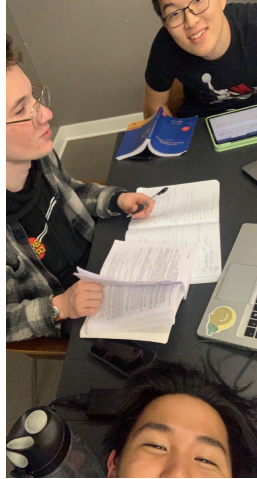# Northwestern University

Real Analysis

MATH 321-1

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theorems! definitions! examples!

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14 December 2022

1   The Real and Complex Numbers

Prototypical example for this section: $\mathbb{R}$ *is an ordered field with the least-upper-bound property and subfield* $\mathbb{Q}$.

Definition 1.1. Let S be any set.

- If x is a member of S, we write $x \in S$. Otherwise, we write $x \notin S$.

- The set without any elements is called the **empty set**, often written as $\emptyset$.

- If A and B are sets and every element of A is an element of B, we say that A is a *subset* of B and write $A \subset B$ or $B \supset A$.

- If $A \subset B$ and $B \subset A$, we write $A = B$.

Definition 1.2. Let S be a set. An **order** on S is a relation, $<$, with the following two properties:

1. If $x, y \in S$, then *exactly* one of the statements is true:
$$x < y, \quad x = y, \quad y < x$$

2. If $x, y, z \in S$, if $x < y$, and $y < z$, then $x < z$. This is called **transitivity**.

It's then natural to define an **ordered set** to be a set S in which an order is defined.

Example 1.3 (Ordered Sets)

- $\mathbb{Q}$ is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

- $\mathbb{C}$ is not ordered.

Definition 1.4. Suppose S is ordered and $E \subset S$. E is *bounded above* if there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$. We call β an *upper bound* of E.

Furthermore, if any element $\gamma < \beta$ is not an upper bound of E, we call β the **least upper bound** or **supremum** of E and write $\beta = \sup E$. (The *greatest lower bound*, or *infimum* of E is defined similarly).

Example 1.5 (Upper and Lower Bounds)

1. Let A be the set of all positive rationals p such that $p^2 < 2$ and B be the set of all positive rationals q such that $q^2 > 2$. The *upper bounds* of A are exactly the elements of B. Since B contains no smallest element, A does *not* have a least upper bound.

2. Let E_1 be the set of all $r \in \mathbb{Q}$ with $r < 0$. Let E_2 be the set of all $r \in \mathbb{Q}$ with $r \leq 0$. Then
$$\sup E_1 = \sup E_2 = 0,$$
and notice $0 \notin E_1$, $0 \in E_2$.

3. Let $E = \{1/n \mid n \in \mathbb{N}\}$. Then $\sup E = 1 \in E$ but $\inf E = 0 \notin E$.

Definition 1.6. Let S be an ordered set. If any nonempty and bounded above $E \subset S$ has a *least upper bound* in S, then S is said to have the **Least Upper Bound Property**

Example 1.7 (Least Upper Bound Property)

1. $\mathbb{Q}$ does not have the Least Upper Bound Property (Example 1.5.1)

2. $\mathbb{R}$ does have the Least Upper Bound Property.

Theorem 1.8

Suppose S is an ordered set with the Least Upper Bound Property, $B \subset S$, $B \neq \emptyset$, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L = \inf B$$

exists in S.

Definition 1.9. A set F with two operations, called *addition* and *multiplication*, is called a *field* if the following "field axioms" are satisfied:

1. Axioms for Addition

    (a) Closure: $x \in F, y \in F \Rightarrow x + y \in F$.
    (b) Commutativity: $x + y = y + x$ for all $x, y \in F$.
    (c) Associativity: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
    (d) Identity: F contains an element 0 such that $0 + x = x$ for every $x \in F$.
    (e) Inverse: For every $x \in F$ there exists a $-x \in F$ such that x+(-x)=0.

2. Axioms for Multiplication

    (a) Closure: $x \in F, y \in F \Rightarrow xy \in F$;;;;;;;.
    (b) Commutativity: $xy = yx$ for all $x, y \in F$.
    (c) Associativity: $x(yz) = (xy)z$ for all $x, y, z \in F$.
    (d) Identity: F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
    (e) Inverse; For every nonzero $x \in F$, there exists an element $1/x \in F$ such that $x \cdot (1/x) = 1$.

3. The Distributive Law
$$x(y + z) = xy + xz$$
    holds for all $x, y, z \in F$.

Example 1.10 (Fields)

1. The set of rationals $\mathbb{Q}$ is a field with customary operations.

2. The set of integers $\mathbb{Z}$ is *not* a field with standard addition and multiplication. (Why?)

Proposition 1.11 (The axioms for addition imply the following statements)

1. If $x + y = x + z$, then $y = z$.

2. If $x + y = x$ then $y = 0$.

3. If $x + y = 0$, then $y = -x$.

4. $-(-x) = x$.

Naturally, similar statements arise from the multiplication axioms.

Proposition 1.12 (The axioms for multiplication imply the following statements)

1. If $x \neq 0$ and $xy = xz$, then $y = z$.

2. If $x \neq 0$ and $xy = x$, then $y = 1$.

3. If $x \neq 0$ and $xy = 1$, then $y = 1/x$.

4. If $x \neq 0$, then $1/(1/x) = x$.

Furthermore, we can formulate statements about the combined behavior of addition and multiplication.

Proposition 1.13 (The field axioms imply the following statements, for any $x, y, z \in F$)

1. 0x=0

2. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

3. $(-x)y = -(xy) = x(-y)$.

4. $(-x)(-y) = xy$.

Definition 1.14. An **ordered field** is a *field F* which is also an *ordered set*, such that

1. If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

2. If $x, y \in F$ and $x, y > 0$, then $xy > 0$.

Example 1.15 (Ordered Fields)

1. $\mathbb{Q}$ is an ordered field.

2. $\mathbb{C}$ is *not* an ordered field with the usual notions of addition and multiplication. *But it can be ordered if we get creative.* Consider

$$(a + bi) \leq (c + di) \iff (a < c) \vee [(a = c) \wedge (b \leq d)]$$

All the familiar rules of working with inequalities apply in every ordered field:

Proposition 1.16 (The following statements are true in every ordered field.)

1. If $x > 0$, then $-x < 0$ (and vice versa).

2. If $x > 0$ and $y < z$, then $xy < xz$.

3. If $x < 0$ and $y < z$, then $xy > xz$.

4. If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.

5. If $0 < x < y$, then $0 < 1/y < 1/x$.

1.1   The Real Field

The following existence theorem is the core of this entire chapter:

> **Theorem 1.17** (The Existence of $\mathbb{R}$)
>
> There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

In other words, $\mathbb{Q} \subset \mathbb{R}$ and the operations of addition and multiplication in $\mathbb{R}$ also apply to $\mathbb{Q}$.

> **Theorem 1.18** (The Archimedian Principle and Density of $\mathbb{Q}$)
>
> 1. If $x, y \in \mathbb{R}$, and $x > 0$, then there exists a positive integer $n \in \mathbb{N}^+$ such that
>
> $$nx > y.$$
>
> 2. If $x, y \in \mathbb{R}$ and $x < y$, then there exists a rational $p \in \mathbb{Q}$ such that $x < p < y$.

Remark 1.19. To prove (1), consider the set $\{nx \mid n \in \ltimes\}$ and suppose that it's bounded above by y. It has a supremum α. (Why?) A contradiction arises when considering $\alpha - x$. To prove (2), use (1) on $y - x > 0$ to obtain a $n \in \mathbb{N}$ such that $n(y - x) > 1$. Then use (1) again to obtain $m_1, m_2 \in \mathbb{N}$ such that $-m_2 < nx < m_1$. Then there exists an $m \in \mathbb{N}$ such that $m - 1 \leq nx < m$. Combining existing inequalities and noting $n > 0$, we get $x < \frac{m}{n} < y$.

We can now prove the existence of nth roots of positive real numbers.

> **Theorem 1.20**
>
> For every real $x > 0$ and every integer $n > 0$, there exists *exactly* one positive real y such that $y^n = x$. This number y is written as $\sqrt[n]{x}$ or $x^{1/n}$.

Remark 1.21. $0 < y_1 < y_2$ implies $y_1^n < y_2^n$, so the constraint of there being *at most* one y is clear. Consider the set $E = \{t \in \mathbb{R} \mid t^n < x\}$. Notice E is nonempty since $(\frac{x}{1+x})^n \leq \frac{x}{1+x} < x$. Thus, $y = \sup E$ exists. Use the inequality $b^n - a^n < (b - a)nb^{n-1}$ that holds when $0 < a < b$ to show $y^n < x$ and $y^n > x$ lead to contradictions.

> **Corollary 1.22**
>
> If a and b are positive real numbers and n is a positive integer, then
>
> $$(ab)^{1/n} = a^{1/n}b^{1/n}.$$

1.2   The Extended Real Numbers

Definition 1.23. The *extended real number system* consists of the real field $\mathbb{R}$ and two symbols: $+\infty$ and $-\infty$, with original order of $\mathbb{R}$ preserved and

$$-\infty < x < +\infty$$

defined for every $x \in \mathbb{R}$.

It is customary to make the following conventions:

1. If $x \in \mathbb{R}$, then
$$x + \infty = +\infty, \quad x - \infty = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

2. If $x > 0$, then $x \cdot (+\infty) = +\infty, \quad x \cdot (-\infty) = -\infty.$

3. If $x < 0$, then $x \cdot (+\infty) = -\infty, \quad x \cdot (-\infty) = +\infty.$

1.3   The Complex Field

Definition 1.24. A **complex number** is an *ordered* pair (a, b) of real numbers. Note that (a, b) and (b, a) are distinct if $a \neq b$. If $x = (a, b), y = (c, d)$ are two complex numbers, note that $x = y$ if, and only if, $a = c$ and $b = d$. We define
$$x + y = (a + c, b + d)$$
$$xy = (ac - bd, ad + bc)$$

Theorem 1.25

The complex numbers are a field, with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1. Notice $(a, 0) + (b, 0) = (a+b, 0)$ and $(a, 0)(b, 0) = (ab, 0)$.

The last result leads to the customary way of representing a complex number (a, b) as $a + bi$.

Definition 1.26. We define
$$i = (0, 1)$$

Theorem 1.27

1. $i^2 = -1$

2. If $a, b \in \mathbb{R}$, then $(a, b) = a + bi$

Definition 1.28. If $z = a + bi \in \mathbb{C}$, then the *conjugate* of z is $\bar{z} = a - bi$. The real and imaginary parts of z are often written
$$a = \Re(z), \quad b = \Im(z)$$

Theorem 1.29

If $z, w \in \mathbb{C}$, then

1. $\overline{z + w} = \bar{z} + \bar{w}$,

2. $\overline{zw} = \bar{z} \cdot \bar{w}$,

3. $z + \bar{z} = 2\Re(z), \quad z - \bar{z} = 2\Im(z)$,

4. $z\bar{z}$ is real and positive (except when $z = 0$).

We can now associate with complex (and real!) numbers a notion of size, known formally as the *absolute value*.

Definition 1.30. If $z \in \mathbb{C}$, then its **absolute value**, $|z|$, is the defined to be $|z| = \sqrt{z\bar{z}}$.

Remark 1.31. Note that if $x \in \mathbb{R}$, then $\bar{x} = x$, so $|x| = \sqrt{x^2}$. Thus

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Theorem 1.32

Let $z, w \in \mathbb{C}$. Then

1. $|z| > 0$ unless $z = 0$, $|0| = 0$m

2. $|\bar{z}| = |z|$,

3. $|zw| = |z||w|$,

4. $|\Re(z)| \leq |z|$,

5. $|z + w| \leq |z| + |w|$. (This is known as the *triangle inequality*.)

Some quick notation: If $x_1, \ldots, x_n \in \mathbb{C}$, we write

$$x_1 + x_2 + \cdots + x_n = \sum_{j=1}^{n} x_j.$$

We will find the following inequality to be very important, in foresight of future chapters:

Theorem 1.33 (Cauchy-Schwartz Inequality)

If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$, then

$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right| \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Remark 1.34. To prove, let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, and $C = \sum a_j \bar{b}_j$. Assume $B > 0$ since the $B = 0$ case is trivial. Show that $\sum |Ba_j - Cb_j|^2 = B(AB - |C|^2)$. Then $AB - |C|^2 \geq 0$.

1.4   The Euclidean Space

Definition 1.35. For $k \in \mathbb{N}$, let $\mathbb{R}^k$ to be the set of all *ordered k-tuples*

$$\vec{x} = (x_1, x_2, \ldots, x_k),$$

where $x_1, \ldots, x_k$ are called the *coordinates* of $\vec{x}$. The elements of $\mathbb{R}^k$ are called *points* or *vectors*. If $\alpha \in \mathbb{R}$ and $\vec{y} = (y_1, \ldots, y_k)$, we defined

$$\vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_k + y_k),$$
$$\alpha\vec{x} = (\alpha x_1, \ldots, \alpha x_k)$$

so $\vec{x} + \vec{y} \in \mathbb{R}^k$ and $\alpha\vec{x} \in \mathbb{R}^k$.

These operations of vector addition and scalar multiplication with a vector make $\mathbb{R}^k$ into a *vector space over* $\mathbb{R}$. We define the **inner product** of $\vec{x}$ and $\vec{y}$ by

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$$

and the **norm** of $\vec{x}$ by

$$|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}$$

The structure of $\mathbb{R}^k$ with inner product and norm is called euclidean k-space.

Theorem 1.36

Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k, \alpha \in \mathbb{R}$. Then

1. $|\vec{x}| \geq 0$;
2. $|\vec{x}| = 0$ if, and only if, $\vec{x} = \vec{0}$;
3. $|\alpha\vec{x}| = |\alpha||\vec{x}|$;
4. $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$;
5. $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$;
6. $|\vec{x} - \vec{z}| \leq |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|$.

Remark 1.37. (a),(b),(c) are obvious, and (d) immediately follows from Cauchy-Schwartz. It follows from (d) that $|\vec{x} + \vec{y}|^2 = (|\vec{x} + \vec{y}|)^2$, proving (e). Finally, (f) immediately follows from (e).

2   Basic Topology

Prototypical example for this section: Sets are not doors! Also, a set in $\mathbb{R}$ is compact if, and only if, it is closed and bounded.

2.1   Finite, Countable, and Uncountable Sets

Definition 2.1. Consider two sets A, B, where each element $x \in A$ has an associated element of B, denoted $f(x)$. Then we say f is a *function* (or *mapping*) from A to B, written

$$f : A \to B.$$

We call A the *domain*, B the *codomain*, and the set $f(A) = \{f(x) \mid x \in A\}$ the *range* of f.

Example 2.2

1. $f(x) = x^2$ is a function from $\mathbb{R}$ to $\mathbb{R}$ with domain $\mathbb{R}$, codomain $\mathbb{R}$, and range $[0, \infty)$.

2. Let $f(x) = x$ for all $x \in A$. Then f (called the *identity mapping*) is a function from with domain A and range A. Note that we don't know the codomain of f based on what's given.

3. The Direchlet function is a type of *indicator function* defined as

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Definition 2.3. Let $f : A \to B$ be a mapping. If $E \subset A$, we call $f(E)$ to be the *image* of E under f. Clearly, $f(A) \subset B$, and if $f(A) = B$, we say f maps A **onto** B.

If $E \subset B$, the *inverse image* of E under f is the set $f^{-1}(E)$ of all $x \in A$ such that $f(x) \in E$. If $y \in B$, then $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If for each $y \in B$, f^{-1} consists of *at most* element of A, we call f a **one-to-one** mapping of A into B. In other words, f is one-to-one (or *injective*) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1, x_2 \in A$.

Example 2.4

1. The identity function $f(x) = x$ is one-to-one and onto.

2. $f(x) = |x|$ is onto $\mathbb{R}^+$, but not one-to-one.

3. The logistic function $S(x) = \frac{1}{1+e^{-x}}$ is one-to-one, but not onto. (Fun fact: it's a sigmoid curve!)

Definition 2.5. If there exists an injective mapping of A onto B, we say A and B are in 1-1 correspondence, or that they have the same cardinal number, or that they are equivalent; and write $A \sim B$. This relation has the following properties:

1. Reflexive: $A \sim A$,

2. Symmetric: If $A \sim B$, then $B \sim A$,

3. Transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

Definition 2.6. For any $n \in \mathbb{N}$, let J_n be the set $\{1, 2, \ldots, n\}$. We say

1. A is *finite* if $A \sim J_n$ for some n. (Note that $\{\emptyset\}$ is finite.)

2. A is *infinite* if A is not finite.

3. A is *countable* if $A \sim \mathbb{N}$.

4. A is *uncountable* if A is neither finite nor countable.

5. A is *at most countable* if A is finite or countable.

For two finite sets A and B, $A \sim B$ if, and only if, A and B have the same number of elements.

Example 2.7

The set of all integers $\mathbb{Z}$ is countable. (For the wary reader, consider the function

$$f(n) = \begin{cases} \frac{n}{2} & (n \text{ even}), \\ -\frac{n-1}{2} & (n \text{ odd}) \end{cases}$$

which sets up 1-1 correspondence between $\mathbb{Z}$ and $\mathbb{N}$.

Remark 2.8. It's obvious, then, that a finite set cannot be equivalent to any of its proper subsets. However, it is possible with infinite sets (we just showed this very case in Example 2.7). This leads to another definition of infinite sets: a set A is infinite if A is equivalent to one of its proper subsets.

Definition 2.9. A function f with domain $\mathbb{N}$ is called a sequence, often written $\{x_n\}$ if $f(n) = x_n$. If A is a set and if $x_n \in A$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a sequence in A, or a sequence of elements in A. Notice that every countable set can be seen as the range of a sequence of distinct terms.

Theorem 2.10

Every infinite subset of a countable set A is countable.

Definition 2.11. Let A and Ω be sets, and suppose that with each element α of A there is an associated subset of Ω, denoted E_α. The **union** of the sets E_α is defined to be the set S such that $x \in S$ if, and only if, $x \in E_\alpha$ for at least one $\alpha \in A$. This is notated by

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

If A consists of the integers $1, 2, \ldots, n$, we write

$$S = \bigcup_{i=1}^{n} E_i = E_1 \cup \cdots \cup E_n.$$

If $A = \mathbb{N}$, we notate the countable union as follows:

$$S = \bigcup_{i=1}^{\infty} E_i.$$

The **intersection** of the sets E_α is defined to be the set P such that $x \in P$ if, and only if, $x \in E_\alpha$ for every $\alpha \in A$. We notate similarly with

$$P = \bigcap_{\alpha \in A} E_\alpha$$

or
$$P = \bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap \cdots \cap E_n$$

or
$$P = \bigcap_{i=1}^{\infty} E_i.$$

If $A \cap B \neq \emptyset$, we say that A and B intersect; otherwise they are *disjoint*.

Example 2.12

Suppose $E_1 = \{1, 2, 3\}$ and $E_2 = \{2, 3, 4\}$. Then $E_1 \cup E_2 = \{1, 2, 3, 4\}$ and $E_1 \cap E_2 = \{2, 3\}$.

Let $A = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$. For every $x \in A$, let $E_x = \{y \in \mathbb{R} \mid 0 < y < x\}$. Then

1. $E_x \subset E_z$ if, and only if, $0 < x \leq z \leq 1$;

2. $\bigcup_{x \in A} E_x = E_1$;

3. $\bigcap_{x \in A} E_x = \emptyset$.

Remark 2.13. Unions and intersections behavior very similarly to sums and products. Naturally, the commutative and associative laws hold:

$$A \cup B = B \cup A \qquad\qquad\qquad A \cap B = B \cap A$$
$$(A \cup B) \cup C = A \cup (B \cup C) \qquad\qquad (A \cap B) \cap C = A \cap (B \cap C)$$

The distributive law also holds:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Finally, the following relations hold:

1. $A \subset A \cup B$,
2. $A \cap B \subset A$,
3. $A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset$,
4. If $A \subset B$, then $A \cup B = B, \qquad A \cap B = A$

Theorem 2.14

The countable union of countable sets is countable.

Remark 2.15. To prove this, consider the countable union $S = \bigcup_{n=1}^{\infty} E_n$, and let every countable set E_n be arranged in a sequence $\{x_{nk}\}$. Then we can arrange the elements of S in a sequence: $x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, \ldots$. (See Rudin pg. 29)

Corollary 2.16

If A is at most countable and for every $\alpha \in A$, B_α is at most countable. Then $T = \bigcup_{\alpha \in A} B_\alpha$ is at most countable.

> **Theorem 2.17**
>
> Let A be a countable set. The set of all n-tuples $\{(a_1, \ldots, a_n) \mid a_k \in A\}$ is countable.

Applying this theorem with $n = 2$ gives an important result:

> **Corollary 2.18**
>
> The set of rational numbers $\mathbb{Q}$ is countable.

> **Theorem 2.19**
>
> The set A of all sequences whose elements are 0 and 1 is uncountable.

Remark 2.20. To prove, take a countable subset E of A, where E is a set of sequences $\{s_1, s_2, \ldots\}$. Then construct a sequence s as follows: the nth term of s is 1 if the nth term of s_n is 0, and vice versa. Then s differs from every element of E in at least one place so $s \notin E$. Clearly, $s \in A$ so E is a proper subset of A. Thus every countable subset of A is proper, so A is uncountable. $\square$

This type of argument is known as Cantor's digaonal process; for sequences $s_1, s_2, \ldots$ are placed in an array, it's the elements on the diagonal involved in the construction of a new sequence.

A direct corollary is that the real numbers $\mathbb{R}$ are uncountable (via the binary representation of real numbers).

2.2   Metric Spaces

Definition 2.21. Let X be a set (whose elements we call *points*) is said to be a **metric space** if there exists a real-valued *distance function*, or *metric*, such that for any $p, q \in X$:

1. $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;

2. $d(p, q) = d(q, p)$;

3. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

> **Example 2.22** (Metric Spaces)
>
> 1. The euclidean spaces $\mathbb{R}^k$, especially $\mathbb{R}$ (the real line) and $\mathbb{R}^2$ (the complex plane), are metric spaces with the **euclidean distance** defined by
>
> $$d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| \quad (\vec{x}, \vec{y} \in \mathbb{R}^k)$$
>
> 2. The **discrete metric** is defined as
>
> $$d(p, q) = \begin{cases} 1 & p = q \\ 0 & p \neq q \end{cases}$$
>
> 3. Subsets of $\mathbb{R}^k$ are metric spaces.

> **Example 2.23** (L^p spaces)
>
> An important example of metric spaces (that we'll see in later chapters) are the L^p spaces. For any real number $p \geq 1$, define L^p to be the set of all (say, real-valued) functions
>
> $$L^p = \{f : A \to \mathbb{R} \quad | \quad \int_A |f(x)|^p \ dx < \infty\}$$
>
> We define the L^p norm to be
>
> $$\|f\|_p = \left(\int_A |f(x)|^p \ dx \right)^{1/p}.$$
>
> And a distance function
>
> $$d_p(f, g) = \|f - g\| = \left(\int_A |f(x) - g(x)|^p \ dx \right)^{1/p}$$
>
> It can be shown that L^p is a metric, as long as functions that are *almost* equal are treated as equal (thus having distance 0). It turns out that the L^p spaces have many nice properties:
>
> - They are **complete** (see chapter 3).
>
> - They are vector spaces.
>
> - They have an inner product, a notion of orthogonality, and Cauchy-Schwartz holds for L^2.

Definition 2.24. In $\mathbb{R}$, we have open intervals (called *segments*):

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\};$$

closed intervals (called *intervals*):

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\};$$

and half-open intervals: $[a, b)$ and $(a, b]$. If $a_i < b_i$ for $i = 1, \ldots, k$, a *k-cell* is the set of all points

$$\{\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k \mid a_i \leq x_i \leq b_i \quad (1 \leq i \leq k)\}.$$

A 1-cell is an interval, a 2-cell a rectangle, and so on. If $\vec{x} \in \mathbb{R}^k$ and $r > 0$, the **open** (or *closed*) **ball** B centered at $\vec{x}$ with radius r is defined to be the set

$$B_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^k \ : \ |\vec{y} - \vec{x}| < r \quad (\text{or } |\vec{y} - \vec{x}| \leq r)\}$$

A set $E \subset \mathbb{R}^k$ is *convex* if $\lambda \vec{x} + (1 - \lambda)\vec{y} \in E$ whenever $\vec{x}, \vec{y} \in E$ and $0 < \lambda < 1$.

> **Example 2.25** (Balls are convex!)
>
> If $|\vec{y} - \vec{x}| < r$, $|\vec{z} - \vec{x}| < r$, and $0 < \lambda < 1$, we have
>
> $$\begin{aligned} |\lambda \vec{y} + (1 - \lambda)\vec{z} - \vec{x}| &= |\lambda(\vec{y} - \vec{x}) + (1 - \lambda)(\vec{z} - \vec{x})| \\ &\leq \lambda|\vec{y} - \vec{x}| + (1 - \lambda)|\vec{z} - \vec{x}| \\ &< \lambda r + (1 - \lambda)r \\ &= r \end{aligned}$$
>
> A similar argument shows that closed balls and k-cells are convex.

Definition 2.26. This one's a doozy: Let X be a metric space with all points and sets below elements and subsets of X.

1. A **neighborhood** N_r of p is the set consisting of all q such that $d(p, q) < r$ for some *radius* $r > 0$.

2. A point p is a **limit point** of the set E if *every* neighborhood of p contains a point $q \neq p$ that is an element of E.

3. A point p in E is called an **isolated point** if p is not a limit point of E.

4. The set E is **closed** if it contains all its limit points.

5. A point p is an **interior point** if p has a neighborhood $N \subset E$.

6. The set E is open if every point of E is an interior point.

7. The *complement* of E, E^c, is the set of all points $p \in X$ such that $p \notin E$.

8. A closed set E is **perfect** if every point of E is a limit point of E.

9. E is **bounded** if there exists some real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$. **Note that q need not be in E!**

10. E is **dense** in X if every point of X is either a limit point of E, a point of E (or both!)

Theorem 2.27

Every neighborhood is an open set.

Remark 2.28. This follows from the Triangle Inequality. Let $E = N_r(p)$ and note that for any $q \in E$, $d(p, q) < r$ is equivalent to saying $d(p, q) = r - h$ for some $h > 0$. We use this h as the radius of some neighborhood of $q \in E$.

Theorem 2.29

If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.

Remark 2.30. If we suppose there is some neighborhood N of p with only finitely many points of E, we can pick and choose from that list of points (we'll call it $q_1, \ldots, q_n$). A contradiction arises when we pick $r = \min_{1 \leq m \leq n} d(p, q_m)$ to be the radius of $N_r(p)$.

Corollary 2.31

A finite set has no limit points.

Theorem 2.32 (DeMorgan)

Let $\{E_\alpha\}$ be a colletion of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha \right)^c = \bigcap_\alpha (E_\alpha^c).$$

Remark 2.33. When proving equality of two sets $A = B$, it's often the cases that a proof proceeds by showing $A \subset B$ and then $B \subset A$. Thus the proof of this theorem consists of showing that if $x \in \left(\bigcup_\alpha E_\alpha \right)^c$, then $x \in \bigcap_\alpha (E_\alpha^c)$ and vice versa.

Theorem 2.34

A set is open if, and only if, its complement is closed.

Remark 2.35. Note that a set can be both open and closed at the same time ($\emptyset$ and $\mathbb{R}$ are the typical examples).

Corollary 2.36

A set is closed if, and only if, its complement is open.

It now seems natural to think about what happens when you take unions and intersections of sets.

Theorem 2.37

1. The arbitrary union of open sets $\bigcup_\alpha E_\alpha$ is open.

2. The arbitrary intersection of closed sets $\bigcap_\alpha E_\alpha$ is closed.

3. The finite intersection of open sets $\bigcap_{i=1}^n E_i$ is open.

4. The finite union of closed sets $\bigcup_{i=1}^n E_i$ is closed.

Example 2.38 (Unions and Intersections of Closed and Open Sets)

Notice that (3) and (4) only apply to the *finite* intersection and union.

1. For any $n \in \mathbb{N}$, the open interval $(-1/n, 1/n)$ is open. However, we know that the intersection (by A.P.)
$$\bigcap_{i=1}^\infty (-1/n, 1/n) = \{0\},$$
is not open.

2. Similarly, the singletons are closed, so each point $1/n$ is closed, where $n \in \mathbb{N}$. However, the set
$$\{\frac{1}{n} \mid n \in \mathbb{N}\}$$
has limit point 0, which is not in the set.

Definition 2.39. Let X be a metric space and $E \subset X$. We denote the set of limit points of E by E', and define the **closure** of E to be $\overline{E} = E \cup E'$.

Obviously, the closure of a set contains the set itself and it seems natural that the closure of a set must be closed itself.

Theorem 2.40

If X is a metric space and $E \subset X$, then

1. $\overline{E}$ is closed.

2. $E = \overline{E}$ if, and only if, E is closed.

3. $\overline{E} \subset F$ for every closed set $F \supset E$.

By (1) and (3), the closure of E is the *smallest* closed set containing E.

Remark 2.41. The proofs of (2) and (3) are trivial, but it is worth noting that Rudin proves (1) by showing that the complement of $\overline{E}$ is open. This approach can be very useful when showing a set is closed.

Theorem 2.42

Let E be a nonempty set of real numbers that is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$; thus $y \in E$ if E is closed.

So far, the notions of open and closed exist in a global sense (in say, a metric space X). But now consider the case when $E \subset Y \subset X$. We know that *subspaces of metric spaces are themselves metric spaces*, so Y is also a metric space.

Now, say E is open in X. From the definition of a set being open, for every $p \in E$, there exists some real number $r > 0$ such that a neighborhood around p of radius r is completely contained in E, i.e. $d(p, q) < r$ implies $q \in E$.

It seems perfectly reasonable to now consider the case when E is open in Y, but not necessarily in X. We say E is **open relative** to Y if for every $p \in E$, there exists some $r > 0$ such that $d(p, q) < r$ implies $p \in Y$.

Example 2.43

1. The open interval (a, b), where $a, b \in \mathbb{R}, a < b$ is open relative to $\mathbb{R}$, but not open in $\mathbb{R}^2$.

2. The half-open interval $(0, 1]$ is open relative to $[-1, 1]$ but not open in $\mathbb{R}$.

From these examples, another way of thinking about relative openness seems to creep around the corner..

Theorem 2.44

Suppose $Y \subset X$. A subset $E \subset Y$ is open relative to Y if, and only if, there exists some set $G \subset X$ that is open in X such that
$$E = Y \cap G.$$

Remark 2.45. To prove the forward direction, we use the definition of relative openness to obtain a r_p for every point $p \in E$, let $V_p = \{q \in X \mid d(p, q) < r_p\}$ and let G be the union of open neighborhoods $G = \bigcup_p V_p$. We then show that $E \subset Y \cap G$ and $Y \cap G \subset E$. The backwards direction just consists of checking the conditions from the definition of relative openness of a set.

2.3   Compactness

Definition 2.46. Let E be a subset of a metric space X. An open cover of E is a collection $\{G_\alpha\}$ of open subsets of X such that

$$E \subset \bigcup_\alpha G_\alpha.$$

Definition 2.47. A subset K of a metric space X is said to be **compact** if every open cover of K has a finite subcover.

In other words, if $\{G_\alpha\}$ is an open cover of K, there are finitely many indices $i_1, \ldots, i_n$ such that

$$\bigcup_{j=1}^n G_{i_j} \supset K.$$

We have seen so far that the notions of open and closed depend on the space in which a set is embedded. However, as we'll soon see, compactness works differently; compactness depends only on the set itself, not any embedding space.

Theorem 2.48

Suppose $K \subset Y \subset X$. Then K is compact relative to X if, and only if, K is compact relative to Y (where a set being relatively compact has a similar definition to it being relatively open).

Remark 2.49. By this theorem, we are able to regard compact spaces in their own right without further context.

Theorem 2.50

Compact spaces are closed.

Remark 2.51. The proof of this theorem proceeds by showing the complement of a compact space is open. Let K be compact, $p \in K^c$, and $q \in K$. Let V_q and W_q be neighborhoods of p and q, respectively, each with radius less than $\frac{1}{2}d(p,q)$. Now, because K is compact, there exist $q_1, \ldots, q_n$ such that $\bigcup_{i=1}^n W_{q_i}$ is an open cover of K. Now, consider the neighborhoods $\{V_{q_i}\}$, where each V_{q_i} is a neighborhood of p with radius less than $d(p, q_i)$. Then $\bigcap_{i=1}^n V_{q_i} \subset K^c$ is a neighborhood of p contained entirely in K^c. So p is an interior point and thus K^c is open. (See Figure 1)

Theorem 2.52

Closed subsets of compact sets are compact.

Remark 2.53. Let E be a closed subset of a compact space K. Note: K being closed is a necessary condition! Let $\{V_\alpha\}$ be an open cover of F. Since F^c is open, it follows that $\{V_\alpha\} \cup F^c$ is an open cover of K. Since K is compact, we can obtain a finite subcover $\bigcup_{j=1}^n V_{i_j} \cup F^c$ of K. Since every element of F is obviously not in F^c, it follows that $\bigcup_{j=1}^n V_{i_j}$ is an open cover of F.

Corollary 2.54

If F is closed and K compact, then $F \cap K$ is compact.

We will now begin to build machinery that is not only useful in its right, but also works to formulate a very important result of this chapter.
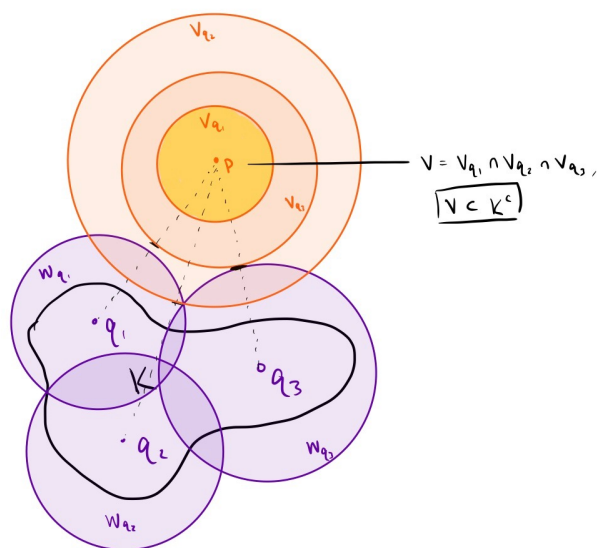
Figure 1: Compact sets are closed.

Theorem 2.55 (Finite Intersection Theorem)

If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that any intersection of finitely many elements of $\{K_\alpha\}$ is nonempty, then the arbitrary intersection $\bigcap_\alpha K_\alpha$ is also nonempty.

Remark 2.56. To prove, fix a member K_1 of K_α and for each K_α, let $G_\alpha = K_\alpha^c$. Assume, for sake of contradiction, that no single point of K_1 belongs to every K_α. Then the collection $\{G_\alpha\}$ forms an open cover of the compact set K_1, so we can find a finite subcover $\{G_{\alpha_i}\}_{i=1}^n$ of K_1. But this means

$$K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}$$

is empty, a contradiction!

Corollary 2.57 (Nested Intersection Theorem)

If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ for $n = 1, 2, \ldots$, then $\bigcap_{i=1}^\infty K_i$ is nonempty.

Theorem 2.58

Every infinite subset of a compact set K has a limit point in K.

Remark 2.59. To prove, suppose that $E \subset K$ does not have any limit points in itself. Then every point in E has a neighborhood with exactly one element in E. The collection of those neighborhoods is an open cover for E, but obviously does not have a finite subcover.

Theorem 2.60 (Lemmas for Heine-Borel)

1. If $\{I_n\}$ is a sequence of (closed) intervals in $\mathbb{R}$ such that $I_n \supset i_{n+1}$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

2. Let $k \in \mathbb{N}$. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset i_{n+1}$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

3. Every k-cell is compact.

Theorem 2.61 (Heine-Borel Theorem)

Let E in $\mathbb{R}^k$. Then the following are equivalent:

1. E is closed and bounded.

2. E is compact.

3. Every infinite subset of E has a limit point in E.

Remark 2.62. It should be noted that (2) and (3) are equivalent in *every* metric space, but (1) does not, in general, imply (2) and (3).

Theorem 2.63 (Weierstrass)

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Remark 2.64. If E is a bounded infinite subset of $\mathbb{R}^k$, then it is an infinite subset of a k-cell, which is compact. Thus E has a limit point in that k-cell.

2.4   Perfect Sets

Theorem 2.65

Every nonempty perfect set in $\mathbb{R}^k$ is uncountable.

Remark 2.66. The perfect set P has limit points, and is thus infinite. If P were to be countable, we can enumerate the points $x_1, x_2, \ldots$ and inductively construct the following sequence of neighborhoods $\{V_n\}$:

1. V_1 is any neighborhood of x_1.

2. Suppose V_n was constructed so that $V_n \cap P \neq \emptyset$. Every point of P is a limit point, so there exists a neighborhood V_{n+1} such that

   (a) $\overline{V_{n+1}} \subset V_n$,
   (b) $x_n \notin \overline{V_n}$,
   (c) $V_{n+1} \cap P \neq \emptyset$.

3. We can continue with our construction, as (c) satisfies the induction hypothesis.

Let $K_n = \overline{V_n} \cap P$. $\overline{V_n}$ is closed and bounded, thus compact. By our construction, $x_n \notin \overline{V_{n+1}}$, so there does not exist a point $p \in K$ such that $p \in \bigcap_{n=1}^{\infty} K_n$. But $K_n \subset P$, so the intersection $\bigcap_{n=1}^{\infty}$ is empty. But each K_n is nonempty by (c) and $K_n \supset K_{n+1}$ by (a), so we have a contradiction with Corollary 2.57.

> **Corollary 2.67**
>
> Every open and closed interval in $\mathbb{R}$ is uncountable. In particular, the set of all real numbers is uncountable.

Example 2.68 (The Cantor Set)

We will construct a perfect set in $\mathbb{R}$ that does not contain any open intervals:

1. Let $E_0 = [0, 1]$ and remove the segment $(\frac{1}{3}, \frac{2}{3})$. Then let

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

2. Remove the middle thirds of these intervals, and let E_2 be the union

$$\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

3. We continue this construction to obtain a sequence of compact sets E_n such that

   (a) $E_1 \supset E_2 \supset E_3 \supset \cdots$

   (b) E_n is the union of 2^{-n} intervals, each of length 3^{-n}.

The set $P = \bigcup_{n=1}^{\infty} E_n$ is called the **Cantor Set**; P is clearly compact, and nonempty by the Finite Intersection Theorem.

- Notice that

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \cap P = \emptyset,$$

  where $k, m \in \mathbb{N}$. Notice that every open interval (α, β) contains a segment of the form $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ with $3^{-m} < \frac{\beta - \alpha}{6}$, so P does not contain any open intervals.

- To show P is perfect, let $x \in P$, S be any neighborhood of X and I_n the interval of E_n containing x. We can choose n large enough that $I_n \subset S$ (remember that I_n gets smaller as n increases), and let x_n be the endpoint of I_n such that $x \neq x_n$. We know from the construction of P that $x_n \in P$, and thus x is a limit point of P!

An interesting property of the Cantor set is that it is an example of an uncountable set that has *measure zero*.

2.5   Connected Sets

Definition 2.69. Two subsets A, B of a metric space X are **separated** if both

$$A \cap \overline{B} = \emptyset \quad \text{and} \quad \overline{A} \cap B = \emptyset.$$

A set $E \subset X$ is **connected** if E is *not* a union of two nonempty separated sets.

Remark 2.70. Obviously, separated sets are disjoint, but not every pair of disjoint set is separated! An illustrative example is as follows: $[0,1]$ and $(1,2)$ are *not* separated since 1 is a limit point of $(1,2)$, but $(0,1)$ and $(1,2)$ *are* separated.

Theorem 2.71

A subset E of $\mathbb{R}$ is connected if, and only if, for any $x, y \in E$ with $x < z < y$, it must follow that $z \in E$.

3   Sequences and Series

Prototypical example for this section: Alternative harmonic series. What about it?

3.1   Convergent Sequences

Definition 3.1. A sequence $\{p_n\}$ in a metric space X is said to **converge** to a point p if for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $d(p_n, p) < \epsilon$ for all $n \geq N$.

In this case, we also say p is the limit of $\{p_n\}$, $p_n \to p$, or $\lim_{n \to \infty} p_n = p$.

If $\{p_n\}$ does not converge, it is said to **diverge**.

> **Remark 3.2.** The notion of convergence is dependent on the embedding space of the sequence. For example, $\{1/n \mid n \in \mathbb{N}\}$ converges in $\mathbb{R}$ but not $(0, \infty)$.

The set of all point $\{p_n\}$ is called the *range* of $\{p_n\}$. If the range is bounded, was say the sequence itself is *bounded*.

Example 3.3

1. If $s_n = 1/n$, then $\lim_{n \to \infty} s_n = 0$. The range is infinite, and the sequence is bounded.

2. If $s_n = n^2$, then the sequence $\{s_n\}$ is unbounded, is divergent, and has infinite range.

3. If $s_n = 1 + \frac{(-1)^n}{n}$, the sequence converges to 1, is bounded, and has infinite range.

4. If $s_n = i^n$, the sequence $\{s_n\}$ is divergent, is bounded, and has finite range.

5. If $s_n = 1$, then $\{s_n\}$ converges to 1, is bounded, and has finite range.

Theorem 3.4

Let $\{p_n\}$ be a sequence in a metric space X.

1. **Tail phenomenon**: $\{p_n\}$ converges to $p \in X$ if, and only if, every neighborhood of p contains p_n for all but finitely many n.

2. **Uniqueness of limits**: If $p, p' \in X$, and if $\{p_n\}$ converges to p and p', then $p' = p$.

3. **Convergent sequences are bounded**: If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

4. **Limit points are the limit of a sequence**: If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n \to \infty} p_n$.

> **Remark 3.5.** The proofs proceed as follows:
>
> 1. If $p_n \to p$, then a neighborhood around p of radius r contains all but at most N elements of $\{p_n\}$. Assuming the converse, then given some $\epsilon > 0$, we know the neighborhood of radius ϵ contains all but finitely many n elements of p_n, so we choose $N \geq n$.
>
> 2. For all $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $d(p_n, p_1) < \frac{\epsilon}{2}$ and $d(p_n, p_2) < \frac{\epsilon}{2}$. So
>
> $$d(p_1, p_2) \leq d(p_1, p_n) + d(p_n, p_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

3. We know that there exists $N \in \mathbb{N}$ such that $d(p_n, p) < 1$ for all $n \geq N$. Letting $r = \max\{1, d(p_1, p), \ldots, d(p_N, p)\}$, we have that $d(p_n, p) < r$ for all $n \in \mathbb{N}$.

4. For every $n \in \mathbb{N}$, there is a point $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$. Given $\epsilon > 0$, choose N such that $N > \frac{1}{\epsilon}$. If $n > N$, then $d(p_n, p) < \epsilon$. **Note**: this technique of choosing p_n such that $d(p_n, p) < \frac{1}{n}$ is very useful.

Theorem 3.6

Suppose $\{s_n\}, \{t_n\}$ are complex sequences, and $\lim_{n\to} s_n = s, \lim_{n\to\infty} t_n = t$. Then

1. $\lim_{n\to\infty}(s_n + t_n) = s + t$;

2. $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty}(c + s_n) = c + s_n$, for any number c.

3. $\lim_{n\to\infty} s_n t_n = st$;

4. $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s \neq 0$.

Theorem 3.7

1. Suppose $\vec{x_n} \in \mathbb{R}^k$ and $\vec{x_n} = (\alpha_{1,n}, \ldots, \alpha_{k,n})$. Then $\{\vec{x_n}\}$ converges to $\vec{x} = (\alpha_1, \ldots, \alpha_k)$ if, and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k).$$

2. Suppose $\{\vec{x_n}\}, \{\vec{y_n}\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence of real numbers, and $\vec{x_n} \to \vec{x}$, $\vec{y_n} \to \vec{y_n}$, and $\beta_n \to \beta$. Then

$$\lim_{n\to\infty}(\vec{x_n} + \vec{y_n}) = \vec{x} + \vec{y}, \quad \lim_{n\to\infty} \vec{x_n} \cdot \vec{y_n} = \vec{x} \cdot \vec{y}, \quad \lim_{n\to\infty} \beta_n \vec{x_n} = \beta\vec{x}.$$

Remark 3.8. The forward direction of (1) follows directly from the definition of the norm. Proving the converse, for each $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|\alpha_{j,n} - \alpha_j| < \frac{\epsilon}{\sqrt{k}}$$

for $(1 \leq j \leq k)$. Thus $n \geq N$ implies

$$|\vec{x_n} - \vec{x}| = \left\{\sum_{j=1}^k |\alpha_{j,n} - \alpha_j|^2\right\}^{1/2} < \epsilon.$$

(2) follows immediately from (1) and Theorem 3.6.

3.2   Subsequences

Definition 3.9. Given a sequence $\{p_n\}$, consider the increasing sequence $\{n_k\}$ of positive integers. The sequence $\{p_{n_k}\}$ is called a **subsequence** of $\{p_n\}$. If $\{p_{n_k}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

Remark 3.10. Obviously, $\{p_n\}$ converges to p if, and only if, every subsequence of $\{p_n\}$ converges to p. Note that $\{p_n\}$ is a subsequence of itself.

> **Theorem 3.11** (Subsequences and Compactness)
>
> 1. A sequence in a compact metric space X has a subsequence that converges to a point in X.
>
> 2. Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Remark 3.12. The proof of (1) is trivial when the range of $\{p_n\}$ is finite. Otherwise, the range of $\{p_n\}$ is an infinite subset of a compact set, and thus has a limit point p. Since every neighborhood of p contains infinitely many points in the range of $\{p_n\}$, we can choose $n_1 \in \mathbb{N}$ such that $d(p_{n_1}, p) < 1$, and $n_i > n_{i-1}$ such that $d(p_{n_i}, p) < \frac{1}{i}$. (This argument should look familiar!) Moreover, (2) follows immediately from (1).

> **Theorem 3.13**
>
> The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Remark 3.14. The proof of this theorem involves showing that a limit point q of E^*, the set of all subsequential limits of a sequence $\{p_n\}$, is contained in E^*. We first start by choosing some $n_1 \in \mathbb{N}$ such that $p_{n_1} \neq q$, and let $\delta = d(q, p_{n_1})$. If we can't find one, we're done. Assume inductively that $n_1, \dots, n_{i-1}$ are already chosen. Because q is a limit point of E^*, there exists some $x \in E^*$ such that $d(x, q) < 2^{-i}\delta$. Since x is a subsequential limit of $\{p_n\}$, there exists an $n_i > n_{i-1}$ such that $d(x, p_{n_i}) < 2^{-i}\delta$. Then

$$d(q, p_{n_i}) \leq d(q, x) + d(x, p_{n_i}) < (2^{-i} + 2^{-i})\delta$$

for $i \in \mathbb{N}$. (This argument should look *really* familiar!)

3.3   Cauchy Sequences

Definition 3.15. A sequence $\{p_n\}$ in a metric space X is **Cauchy** if for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $d(p_n, p_m) < \epsilon$ for all $n, m \geq N$.

> **Example 3.16**
>
> $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is Cauchy. It is convergent in $\mathbb{R}$ but *not* convergent in $(0, \infty)$.
>
> $\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \dots\}$, where each successive term has one more digit of π is Cauchy. It is convergent in $\mathbb{R}$ but *not* convergent in $\mathbb{Q}$.

Definition 3.17. Let E be a nonempty subset of a metric space X, and let

$$S = \{d(p, q) \mid p, q \in E\}.$$

The supremum of S is called the **diameter** of E.

Remark 3.18. If $\{p_n\}$ is a sequence in X and $E = \{p_N, p_{N+1}, \dots\}$, then it's clear that $\{p_n\}$ is a Cauchy sequence if, and only if, $\lim_{N \to \infty} \operatorname{diam} E_N = 0$.

Theorem 3.19

1. diam $\overline{E}$ = diam E.

2. If $\{K_n\}$ is a sequence of compact sets such that $K_n \supset K_{n+1}$ for all $n \in \mathbb{N}$, and if

$$\lim_{n\to\infty} \operatorname{diam} K_n = 0,$$

then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

Remark 3.20. Proving (1) consists of showing diam $E \leq$ diam $\overline{E}$ and diam $\overline{E} \leq$ diam E. Proving (2) is immediate from the Finite Intersection Theorem and a quick contradiction when assuming that the intersection has more than one point.

This machinery allows us to formalize an important intuition about Cauchy and convergent sequences: a limit is explicitly involved in the latter, but not the former.

Theorem 3.21

1. Every convergent sequence is Cauchy.

2. Every Cauchy sequence converges in a *compact* metric space.

3. In $\mathbb{R}^k$, every Cauchy sequence converges.

Definition 3.22. A metric space is said to be **complete** if every Cauchy sequence in it converges.

Example 3.23

1. All compact and Euclidean metric spaces are complete.

2. Every closed subset of a complete metric space is complete.

3. $\mathbb{Q}$ is not complete with $d(x, y) = |x - y|$.

Definition 3.24. A sequence $\{s_n\}$ of real numbers is said to be

1. **monotonically increasing** if $s_n \leq s_{n+1}$ for $n \in \mathbb{N}$;

2. **monotonically decreasing** if $s_n \geq s_{n+1}$ for $n \in \mathbb{N}$.

Theorem 3.25 (Monotone Convergence Theorem)

A monotonic sequence converges if, and only if, it is bounded.

Remark 3.26. Most of the proof is immediate. Suppose $s_n \leq s_{n+1}$ and $\{s_n\}$ is bounded. Then we can let s be the least upper bound of $\{s_n\}$. Because $s = \sup\{s_n\}$, for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$s - \epsilon < s_N \leq s.$$

(Otherwise, $s - \epsilon$ would be an upper bound.) Then for all $n \geq N$, we have that $s - \epsilon < s_n \leq s$.

3.4   Upper and Lower Limits

Definition 3.27. If $\{s_n\}$ is a sequence of real numbers such that for every $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n \geq M$, then we write

$$s_n \to +\infty.$$

A similar definition applies for limits to $-\infty$.

Definition 3.28. Let $\{s_n\} \in \mathbb{R}$ and $E = \{x \in \mathbb{R} \mid s_{n_k} \to x\}$. The set E contains all subsequential limits and possibly $+\infty, -\infty$. The **upper** and **lower limits** of $\{s_n\}$ are defined as follows:

$$\limsup_{n \to \infty} s_n = \sup E, \qquad \liminf_{n \to \infty} s_n = \inf E.$$

Theorem 3.29

Let $\{s_n\}$ be a sequence of real numbers and E be defined as above. Then $\limsup s_n$ has the following two properties:

1. $\limsup s_n \in E$,

2. If $x > \limsup s_n$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n < x$.

Of course, $\liminf$ has similar properties.

Example 3.30

1. Let $\{s_n\}$ be a sequence of all the rationals (we can obtain such a sequence because $\mathbb{Q}$ is countable). Then every real number is a subsequential limit and $\limsup_{n \to \infty} s_n = +\infty$, $\liminf_{n \to \infty} s_n = -\infty$.

2. Let $s_n = \frac{(-1)^n}{1+1/n}$. Then $\limsup_{n \to \infty} s_n = 1$, and $\liminf_{n \to \infty} s_n = -1$.

3. For $\{s_n\} \in \mathbb{R}$, it's the case that $\lim_{n \to \infty} s_n = s$ if, and only if

$$\limsup_{n \to \infty} s_n = \liminf_{n \to \infty} s_n = s.$$

Theorem 3.31

If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n,$$
$$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n$$

> **Example 3.32**
>
> If $p > 0$, then $\lim_{n \to \infty} \frac{1}{n^p} = 0$.
>
> If $p > 0$, then $\lim_{n \to \infty} \sqrt[n]{p} = 1$.
>
> $\lim_{n \to \infty} \sqrt[n]{n} = 1$.
>
> If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n \to \infty} \frac{n^\alpha}{(1+p)^n} = 0$.
>
> If $|x| < 1$, then $\lim_{n \to \infty} x^n = 0$.

3.5   Series

Definition 3.33. Given a sequence $\{a_n\}$, the infinite sum $\{a_1 + a_2 + \cdots = \sum_{n=1}^{\infty} a_n$ is an **infinite series** or *series*. The numbers $s_n = \sum_{k=1}^{n} a_k$ are the **partial sums** of the series, and if $\{s_n\}$ converges to s, then the series *converges* and write

$$\sum_{n=1}^{\infty} a_n = s.$$

If $\{s_n\}$ diverges, the series is said to diverge.

The Cauchy Criterion can be restated in the following form:

> **Theorem 3.34**
>
> $\sum a_n$ converges if, and only if, for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that
>
> $$\left| \sum_{k=n}^{m} a_k \right| \leq \epsilon$$
>
> if $m \geq n \geq N$.

In other words:

> **Theorem 3.35** (The Trivial Test)
>
> If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

> **Remark 3.36.** However, $a_n \to 0$ is not a sufficient condition for convergence of $\sum a_n$. For example, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

> **Theorem 3.37** (Comparison Test)
>
> 1. If $|a_n| \leq c_n$ for $n \geq N_0$, where $N_0 \in \mathbb{N}$, and if $\sum c_n$ converges, then $\sum a_n$ converges.
>
> 2. If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ alos diverges.

3.6   Series of Non-negative Terms

Theorem 3.38 (Geometric Series)

If $0 \leq x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \geq 1$, the series diverges.

Theorem 3.39 (Condensation Test)

Suppose $a_1 \geq a_2 \geq \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if, and only if, the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \cdots$$

converges.

Theorem 3.40 (p-series)

$\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 3.41

If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if $p \leq 1$, the series diverges.

3.7   The Number e

Definition 3.42. The number

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Theorem 3.43

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Theorem 3.44

e is irrational.

3.8   The Root and Ratio Tests

Theorem 3.45 (Root Test)

Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then

1. If $\alpha < 1$, $\sum a_n$ converges;

2. $\alpha > 1$, $\sum a_n$ diverges;

3. If $\alpha = 1$, the test is inconclusive.

Theorem 3.46 (Ratio Test)

The series $\sum a_n$

1. converges if $\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$,

2. diverges if $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ for all $n \geq n_0$, where n_0 is a fixed integer.

3. If $\lim a_{n+1}/a_n = 1$, the test is inconclusive.

Example 3.47

The series $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{2^3}{+} \frac{3^3}{+} \cdots$ is shown to be convergent by the root test, but the ratio test is inconclusive.

The same is true for the series $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \cdots$.

Remark 3.48. The ratio test is usually easier to apply than the root test, but the root test has wider *scope*. In other words, whenever the ratio test shows convergence, the root test does too. Whenever the root test is inconclusive, the ratio test is too.

Theorem 3.49

For any sequence $\{c_n\}$ of positive numbers:

1. $\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n}$,

2. $\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$.

3.9   Power Series

Definition 3.50. Given $\{c_n\} \in \mathbb{C}$, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a **power series**, with *coefficients* c_n. The series will converge or diverge based on z; every power series has an associated *circle of convergence*, such that the power series will converge if z is in the interior of the circle and diverge otherwise.

Theorem 3.51

Given the power series $\sum c_n z^k$, put

$$\alpha = \limsup_{n \to \infty} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}.$$

(If $\alpha = 0$, then $R = +\infty$; if $\alpha = +\infty$, then $R = 0$.) $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Remark 3.52. The proof consists of applying the root test to $a_n = c_n z^n$:

$$\limsup_{n \to \infty} \sqrt[n]{|a_n|} = |z| \limsup_{n \to \infty} \sqrt[n]{c_n} = \frac{|z|}{R}.$$

Note that we can also use the ratio test to obtain a radius of convergence, but it might not capture all z values for which the power series converges.

Example 3.53

1. $\sum n^n z^n$ has $R = 0$.

2. $\sum \frac{z^n}{n!}$ has $R = +\infty$. (It is easier to use the ratio test in this example).

3. $\sum \frac{z^n}{n}$ has $R = 1$, and diverges if $z = 1$ but converges for other $|z| = 1$.

4. $\sum z^n$ has $R = 1$. If $|z| = 1$, the series diverges since the limit of z_n is not 0.

5. $\sum \frac{z^n}{n^2}$ has $R = 1$, and converges for all z with $|z| = 1$, by comparison with $1/n^2$.

3.10   Summation By Parts

Theorem 3.54

Let $\{a_n\}, \{b_n\}$ be sequences, and let $A_n = \sum_{k=0}^{n} a_k$ if $n \geq 0$; and let $A_{-1} = 0$. Then if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Remark 3.55. Summation by parts is especially useful when investigating series of the form $\sum a_n b_n$, particularly when $\{b_n\}$ is monotonic.

Theorem 3.56

Suppose

1. The partial sums A_n of $\sum a_n$ form a bounded sequence;

2. $b_0 \geq b_1 \geq b_2 \geq \cdots$;

3. $\lim_{n \to \infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

Theorem 3.57 (Alternating Series Test)
Suppose

1. $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$;

2. $c_{2m-1} \geq 0$, $c_{2m \leq 0}$, for $m \in \mathbb{N}$.

3. $\lim_{n \to \infty} c_n = 0$.

Then $\sum c_n$ converges.

Example 3.58 (Alternating Harmonic Series)
Let $a_n = \frac{(-1)^n}{n}$. Then $\sum a_n$ converges.

Theorem 3.59
Suppose the radius of convergence of $\sum c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots$, and $\lim_{n \to \infty} c_n = 0$. Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

3.11   Absolute Convergence

Definition 3.60. A series $\sum a_n$ **converges absolutely** if the series $\sum |a_n|$ converges.

Remark 3.61. The comparison, root, and ratio tests are really tests for absolute convergence and cannot give any information about non-absolutely convergent series.

3.12   Addition and Multiplication of Series

Theorem 3.62
If $\sum a_n = A$ and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$, and $\sum c a_n = cA$ for any fixed c.

Definition 3.63 (Cauchy). Given $\sum a_n$ and $\sum b_n$, the **Cauchy product** of the two series is

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Remark 3.64. If $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$, and $A_n \to A$, $B_n \to B$, and $C_n \to C$, it is not necessarily the case that $\{C_n\}$ converges to AB.

Example 3.65
The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges, but the Cauchy product of the series with itself does not converge.

> **Theorem 3.66**
>
> Suppose
>
> 1. $\sum_{n=0}^{\infty} a_n$ converges absolutely,
>
> 2. $\sum_{n=0}^{\infty} a_n = A$,
>
> 3. $\sum_{n=0}^{\infty} b_n = B$,
>
> 4. $c = \sum_{k=0}^{n} a_k b_{n-k}$ for $n = 0, 1, \ldots$.
>
> Then
> $$\sum_{n=0}^{\infty} c_n = AB$$

In other words, the product of two convergent series converges to the the product of their individual limits if at least one of the two series converges absolutely.

> **Theorem 3.67**
>
> If the series $\sum a_n$, $\sum b_n$, and c_n converge to A, B, C, and $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$.

Here, no assumptions are made about absolute convergence.

3.13   Rearrangements

Definition 3.68. Let $\{k_n\}$ be a sequence in which every positive integer appears exactly once. Putting $a'_n = a_{k_n}$, we say that $\sum a'_n$ is a **rearrangement** of $\sum a_n$.

> **Example 3.69**
>
> Consider the convergent series
> $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
> and one of its rearrangements
> $$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$
> in which two positive terms are always followed by one negative. If s is the first sum, then $s < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Since $\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0$ for $k \geq 1$, we see that $s'_3 < s'_6 < s'_9 < \cdots$ where s'_n is nth partial sum of the second sum. Thus $\limsup_{n\to\infty} s'_n > s'_3 = \frac{5}{6}$, so the two sums do not converge to the same value (if any).

Theorem 3.70

Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty.$$

Then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that

$$\liminf_{n \to \infty} s'_n = \alpha, \qquad \limsup_{n \to \infty} s'_n = \beta.$$

Theorem 3.71

If $\sum a_n$ is a series of complex numbers which converge absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

4 Continuity

4.1 Limits of Functions

Definition 4.1. Let X and Y be metric spaces, $E \subset X$, $f : E \to Y$, and p is a limit point of E. We write

$$f(x) \to q \text{ as } x \to p, \quad \text{or} \quad \lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \epsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

> **Remark 4.2.** A few notes:
>
> 1. X and Y might have different distance functions, so we denote each as d_X and d_Y to prevent ambiguity.
> 2. It's of course necessary that $p \in X$, but p does *not* need to be in E. In fact, even *if* $p \in E$, it might be the case that $\lim_{x \to p} f(x) \neq f(p)$.

> **Theorem 4.3**
>
> Assume all variables from Definition 4.1 carry the same meaning. Then
>
> $$\lim_{x \to p} f(x) = q$$
>
> if, and only if,
>
> $$\lim_{n \to \infty} f(p_n) = q$$
>
> for every sequence $\{p_n\}$ in E such that $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$.

> **Remark 4.4.** The proof is a simple exercise in symbolic manipulation. For the forward direction, choose a sequence $\{p_n\}$ that converges to p (with $p_n \neq p$), and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that $d(f(x), q) < \epsilon$ if $0 < d(x, p) < \delta$. Furthermore, there exists an $N \in \mathbb{N}$ such that $d(p_n, p) < \delta$, and thus $d(f(p_n), q) < \epsilon$ for every $n \geq N$.
>
> Conversely, we proceed by contraposition, supposing $\lim_{x \to p} f(x) \neq q$. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, there exists an $x \in X$ such that $d(f(x), q) \geq \epsilon$ if $0 < d(x, p) < \delta$. We can now construct a sequence that converges to p as follows:
>
> 1. For $n \in \mathbb{N}$, let $\delta_n = \frac{1}{n}$.
> 2. Pick x_n such that $d(f(x_n), q) \geq \epsilon$ if $0 < d(x_n, p) < \delta_n$.
>
> The sequence $\{x_n\}$ converges to p and $x_n \neq p$, but $\lim_{n \to \infty} f(x_n) \neq q$.

Definition 4.5 (Sums and Products of Functions)**.** For complex functions f, g defined on E, we have that

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x)g(x)$$

and if $f(x) \geq g(x)$ for every $x \in E$, we sometimes write $f \geq g$.

> **Theorem 4.6**
>
> Suppose $E \subset X$, p is a limit point of E, f, g are complex functions on E, and $\lim_{x \to p} f(x) = A$, $\lim_{x \to p} g(x) = B$. Then
>
> 1. $\lim_{x \to p} (f + g)(x) = A + B$;
>
> 2. $\lim_{x \to p} (fg)(x) = AB$;
>
> 3. $\lim_{x \to p} \left(\frac{f}{g} \right)(x) = \frac{A}{B}$ if $B \neq 0$.

4.2   Continuous Functions

Definition 4.7. Suppose X, Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be **continuous at** p if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \epsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

Remark 4.8.   • If f is continuous at *every* point of E, then f is said to be **continuous on** E.
- Also of note is that f must be defined at p in order for it to be continuous at p.
- If p is an isolated point of E, then every function with E as its domain is (vacuously) continuous at p.

> **Theorem 4.9** (Alternate Definition of Continuity at a Point)
>
> Assume that p is a limit point of E. Then f is continuous at p if, and only if,
>
> $$\lim_{x \to p} f(x) = f(p).$$

> **Theorem 4.10** (Composition of Functions)
>
> Suppose X, Y, Z are metric spaces, $E \subset X$, $f : E \to Y$, $g : f(E) \to Z$, and h is defined as
>
> $$h(x) = g(f(x)) \quad (x \in E).$$
>
> If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.
>
> This function h is called the **composition** or the *composite* of f and g. We frequently use the notation $h = g \circ f$.

Remark 4.11. The proof is fairly straightforward from the definition of continuity at a point.

> **Theorem 4.12** (Continuity and Inverse Images)
>
> A mapping f of a metric space X into a metric space Y is continuous on X if, and only if, $f^{-1}(V)$ is open in X for every open set V in Y.

Remark 4.13. This is oftentimes more useful than the first definition of continuity.

To prove, let f be continuous on X and V be an open set in Y. Let $y = f(p) \in V$, and since V open, there exists some neighborhood $N_\epsilon(y) \subset V$. Since f continuous, there exists a $\delta > 0$ such that $d(f(x), f(p)) < \epsilon$ if $d(x, p) < \delta$. Then $x \in f^{-1}(V)$ as soon as $d(x, p) < \delta$.

Conversely, suppose $f^{-1}(V)$ is open in X for an open set $V \in Y$. Choose some $p \in X$ and let $\epsilon > 0$. Notice that $V = \{y \in Y \mid d(f(p), y) < \epsilon\}$ is an open set, so $f^{-1}(V)$ is also open. In other words, there exists a $\delta > 0$ such that $x \in f^{-1}(x)$ (and thus $f(x) \in V$ or $d(f(x), f(p)) < \epsilon$) if $d(x, p) < \delta$.

Corollary 4.14

A mapping f of a metric space X into a metric space Y is continuous if, and only if, $f^{-1}(C)$ is closed in X for every closed set C in Y.

Theorem 4.15 (Sums, Products, and Quotients of Continuous Functions are Continuous)

Let f, g be complex continuous functions on a metric space X. Then $f + g, fg$, and f/g are continuous on X. (Assuming of course that $g \neq 0$ in the last statement.)

Theorem 4.16

1. Let $f_1, f_2, \ldots, f_k$ be real functions on a metric space X, and $\vec{f}$ be a mapping from X into $\mathbb{R}^k$ defined by
$$\vec{f}(x) = (f_1(x), \ldots, f_k(x)) \quad (x \in X);$$
then f is continuous if, and only if, each of the functions $f_1, \ldots, f_k$ is continuous.

2. If $\vec{f}, \vec{g}$ are continuous mappings of X into $\mathbb{R}^k$, then $\vec{f} + \vec{g}$ and $\vec{f} \cdot \vec{g}$ are continuous on X.

Remark 4.17. The functions $f_1, \ldots, f_k$ are called the **components** of f.

Example 4.18

Every polynomial and rational function (given denominator zero, of course) is continuous on $\mathbb{R}^k$.

Remark 4.19. When referring to a function f, we lose nothing from discarding the complement of the domain of f, so we shall talk about continuous mappings of one metric space into another instead of mappings of subsets, and can do so without ambiguity.

4.3   Continuity and Compactness

Definition 4.20. A mapping $\vec{f}$ of a set E into $\mathbb{R}^k$ is said to be **bounded** if there exists $M \in \mathbb{R}$ such that $|\vec{f}| \leq M$ for all $x \in E$.

Theorem 4.21 (Continuous Images of Compact Sets are Compact)

Suppose X is compact, and $f : X \to Y$ is continuous. Then $f(X)$ is compact.

Remark 4.22. Note that $f(f^{-1}(E)) \subset E$, where $E \subset Y$. (If $E \subset X$, then $f^{-1}(f(E)) \supset E$.) To prove the above theorem, let $\{G_\alpha\}$ be an open cover of $f(X)$, use the continuity of f to see that $\{f(G_\alpha)\}$ is an open cover of X, and use the compactness of X to obtain a finite subcover.

We can observe some consequences of Theorem 4.21:

Theorem 4.23

If X is compact and $\vec{f} : X \to \mathbb{R}^k$ is continuous, then $\vec{f}(X)$ is closed and bounded. Thus $\vec{f}$ is bounded.

Theorem 4.24 (Extreme Value Theorem)

Suppose X is compact, $f : X \to \mathbb{R}$, and $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$. Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Remark 4.25. This directly follows from the fact that a closed and bounded sets in $\mathbb{R}$ contains its supremum and infimum.

Theorem 4.26 (Inverses of Continuous Bijections are Continuous)

Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y (in other words, a bijection from X to Y). Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x \quad (x \in X)$$

is a continuous mapping of Y onto X.

Remark 4.27. It suffices to show that $f(V)$ is open if $V \subset X$ is open. Notice that $f(V^c)$ is compact, thus closed, and since f is a bijection, we have $f(V^c) = f(V)^c$ so $f(V)$ is open.

Definition 4.28. Let $f : X \to Y$. We say f is **uniformly continuous** on X if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \epsilon$$

for all $p, q \in X$ for which $d_X(p, q) < \delta$.

Notice there are subtle differences between uniform continuity and continuity:

- Continuity can be defined at a single point, whereas uniform continuity is inherently defined in terms of a set.

- If f is continuous on X, it is possible to find, for each $\epsilon > 0$ and each point $p \in X$, a number $\delta > 0$, which depends on both p *and* ϵ.

- On the other hand, if f is uniformly continuous on X, then it is possible for each $\epsilon > 0$ to find *one* number $\delta > 0$ that will work for *all* points p of X.

- It follows from the logic above that every uniformly continuous function is continuous (but the converse does not necessarily hold.)

Theorem 4.29

Let X be compact, and $f : X \to Y$ be continuous. Then f is uniformly continuous on X.

Remark 4.30. To prove, we have to show that for some $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(p), f(q)) < \epsilon$ if $d(p, q) < \delta$.

Let $\epsilon > 0$, and for each $p \in X$, define $\phi(p)$ such that for $q \in X$,

$$d(p, q) < \phi(p) \Rightarrow d(f(p), f(q)) < \frac{\epsilon}{2}.$$

Now, let

$$J(p) = \{q \in X \mid d(p, q) < \frac{1}{2}\phi(p).\}$$

It should be clear that the collection $\{J_p\}$ is an open cover of the compact set X, so there exist a finite set of points $p_1, \ldots, p_n \in X$ such that

$$X \subset J(p_1) \cup \cdots \cup J(p_n).$$

We can choose $\delta = \frac{1}{2} \min[\phi(p_1), \ldots, \phi(p_n)]$, and because our list is finite, we know $\delta > 0$. (Notice that this is not necessarily the case for an infinite list of points.) Now, let $p, q \in X$ such that $d(p, q) < \delta$. Then there exists some m, with $1 \leq m \leq n$, such that $p \in J(p_m)$, so

$$d(p, p_m) < \frac{1}{2}\phi(p_m).$$

Moreover, notice that

$$d(q, p_m) \leq d(p, q) + d(p, p_m)$$
$$< \delta + \frac{1}{2}\phi(p_m)$$
$$\leq \phi(p_m)$$

Thus $d(f(p), f(p_m)) < \frac{\epsilon}{2}$ and $d(f(q), f(p_m)) < \frac{\epsilon}{2}$. So

$$d(f(p), f(q)) \leq d(f(p), f(p_m)) + d(f(q), f(p_m)) < \epsilon.$$

Theorem 4.31

Let E be a non-compact set in $\mathbb{R}$. Then

1. There exists an unbounded continuous function on E.

2. There exists a continuous and bounded function on E with no maximum.

3. If E is bounded, then there exists a continuous function on E that is not uniformly continuous.

Example 4.32

1. If E is bounded, and there exists a limit point x_0 of E that is not a point of E, then

$$f(x) = \frac{1}{x - x_0} \quad (x \in E)$$

is continuous on E, but unbounded. In fact, if we let $\epsilon > 0$, $\delta > 0$, and choose a point $x \in E$ such that $|x - x_0| < \delta$, we can take t close enough to x_0 so that we can make the difference $|f(t) - f(x)| > \epsilon$, despite $|t - x| < \delta$. Since δ is arbitrary, f is not uniformly continuous on E.

2. The function

$$g(x) = \frac{1}{1 + (x - x_0)^2} \quad (x \in E)$$

is continuous on E and bounded between 0 and 1. Clearly, $\sup_{x \in E} g(x) = 1$, but $g(x) < 1$ for all $x \in E$ so g has no maximum on E.

3. Assume E is unbounded. Then $f(x) = x$ is continuous but unbounded, $h(x) = \frac{x^2}{1+x^2}$ is continuous and bounded with no maximum ($\sup_{x \in E} h(x) = 1$).

4. If we omit boundedness as a condition from (3), then the statement would be false. For example, if $E = \mathbb{Z}$, then every function on E would be uniformly continuous (take $\delta < 1$).

We'll now illustrate with an example that compactness of the domain is also essential in ensuring the inverse mapping of a continuous bijection is continuous.

Example 4.33

Let $X = [0, 2\pi) \subset \mathbb{R}$, $Y = \{\vec{y} \in \mathbb{R}^2 \mid |\vec{y}| = 1\}$. Then define $\vec{f} : X \to Y$ as follows:

$$\vec{f}(t) = (\cos t, \sin t) \quad (0 \le t < 2\pi).$$

We will take for granted that both cosine and sine are continuous and have nice enough periodic properties such that $\vec{f}$ is a bijection.

Notice that f^{-1} is not continuous at $(1, 0) = \vec{f}(0)$, with, of course, the culprit being that X is not compact. (It might be intriguing to see that Y *is* compact!)

4.4   Continuity and Connectedness

Theorem 4.34 (The Image of a Connected Set Under a Continuous Mapping is Connected)

Let $f : X \to Y$ be continuous and $E \subset X$ be connected. Then $f(E)$ is connected.

Remark 4.35. To prove, we will suppose to the contrary that $f(E) = A \cup B$, where A, B are separated. Let

$$G = E \cap f^{-1}(A), \quad H = E \cap f^{-1}(B).$$

Then $E = G \cup H$, since A, B separated. Notice $A \subset \overline{A}$, so $G \subset f^{-1}(\overline{A})$, and since $f^{-1}(\overline{A})$ is closed by the continuity of f, it follows that $\overline{G} \subset f^{-1}(\overline{A})$. Then $f(\overline{G}) \subset \overline{A}$. Notice

$$f(\overline{G}) \cap f(H) \subset \overline{A} \cap B = \emptyset,$$

so $\overline{G} \cap H = \emptyset$, and similarly, $G \cap \overline{H} = \emptyset$. Thus G, H are separated, a clear contradiction!

> **Theorem 4.36** (Intermediate Value Theorem)
>
> Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

Remark 4.37. The converse might seem to be true (that if for any two points $x_1 < x_2$ and for any $c \in (f(x_1), f(x_2))$, there exists an $x \in (x_1, x_2)$ such that $f(x) = c$, then f is continuous), but consider the following:

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

We will show later that this function is not continuous.

4.5   Discontinuities

If x is a point in the domain of the function f at which f is not continuous, we say that f is **discontinuous** at x.

Definition 4.38. Let f be defined on (a, b). Consider any point x such that $a \leq x < b$. We define the *right-hand limit* of f at x as $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$. To obtain the definition of the *left-hand limit* $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x).

Remark 4.39. At any point $x \in (a, b)$, $\lim_{t \to x} f(t)$ exists if, and only if,

$$f(x+) = f(x-) = \lim_{t \to x} f(t).$$

Definition 4.40. Let f be defined on (a, b).

1. If f is discontinuous at x and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the *first kind*, or a **simple discontinuity** at x.

2. Otherwise, f has a discontinuity of the *second kind*.

Remark 4.41. There are two ways a function can have a simple discontinuity:

1. $f(x+) \neq (x-)$,
2. $f(x+) = f(x-) \neq f(x)$.

> **Example 4.42**
>
> Define
> $$f(x) = \begin{cases} 1 & (x \in \mathbb{Q}), \\ 0 & (x \notin \mathbb{Q}). \end{cases}$$
>
> Then f has a discontinuity of the second kind at every point x, since neither one-sided limit exists.

> **Example 4.43**
>
> Define
> $$f(x) = \begin{cases} x & (x \in \mathbb{Q}), \\ 0 & (x \notin \mathbb{Q}). \end{cases}$$
>
> Then f is continuous at $x = 0$ and has a discontinuity of the second kind at every other point.

Example 4.44

Define
$$f(x) = \begin{cases} x+2 & (-3 < x < -2), \\ -x-2 & (-2 \le x < 0), \\ x+2 & (0 \le x < 1). \end{cases}$$

Then f has a simple discontinuity at $x = 0$ and is continuous everywhere else in $(-3, 1)$.

Example 4.45

Define
$$f(x) = \begin{cases} \sin(\frac{1}{x}) & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Since neither $f(0+)$ nor $f(0-)$ exists, f has a discontinuity of the second kind at $x = 0$. We will show later that f is continuous everywhere else.

4.6 Monotonic Functions

Definition 4.46. Let f be real on (a, b). Then f is **montonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \le f(y)$. In other words, f is *non-decreasing*. We can reverse the last inequality to define a *monotonically decreasing* function.

Remark 4.47. A function is *montonic* if it is either monotonically increasing or monotonically decreasing.

Theorem 4.48

Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point $x \in (a, b)$. More precisely,
$$\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t).$$

Furthermore, if $a < x < y < b$, then
$$f(x+) \le f(y-).$$

(Analogous results evidently hold for monotonically decreasing functions.)

Remark 4.49. The proof hearkens back to many ideas from Chapter 1: The set of numbers $f(t)$, where $a < t < x$, is bounded above by $f(x)$, and thus has a least upper bound α. Obviously, $\alpha \le f(x)$. Let $\epsilon > 0$. Since α is a least upper bound, there exists a $\delta > 0$ such that $a < x - \delta < x$ and
$$\alpha - \epsilon < f(x - \delta) \le A \quad (x - \delta < t < x).$$

f being monotonic gives
$$f(x - \delta) \le f(t) \le A \quad (x - \delta < t < x).$$

Thus
$$|f(t) - \alpha| < \epsilon.$$

Thus $f(x-) = \alpha$, and the same reasoning proves the same inequality for the greatest lower bound and $f(+)$.

Notice that if $a < x < y < b$, then

$$f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t) \quad \text{and} \quad f(y-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t).$$

Corollary 4.50

Montonic functions have no discontinuities of the second kind.

This directly implies that every monotonic function is discontinuous at a countable set of points at most.

Theorem 4.51

Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.

Remark 4.52. The proof proceeds by establishing a bijection between the points of discontinuities and the rationals, a countable set: Suppose f is monotonically increasing and let E be the set of points at which f is discontinuous. Then, for every point $x \in E$, we let $r(x) \in \mathbb{Q}$ such that

$$f(x-) < r(x) < f(x+).$$

Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, it follows that $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$.

Note that discontinuities of a monotonic function do not necessarily need to be isolated. In fact, given a countable subset E of (a, b), (which may even be dense!) we can construct a function f monotonic on (a, b), discontinuous at every point of E, and at no other point of (a, b).

Example 4.53 (la fonction)

Arrange the points of E in a sequence $\{x_n\}$. Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n$ converges (notice that the sum also absolutely converges), and define

$$f(x) = \sum_{x_n < x} c_n \quad (a < x < b).$$

If there are no points x_n to the left of x, the sum is empty; thus zero. Then

1. f is monotonically increasing on (a, b);

2. f is discontinuous at every point of E; in fact,

$$f(x_n+) - f(x_n-) = c_n.$$

3. f is continuous at every other point of (a, b).

Notice that $f(x-) = f(x)$ for all $x \in (a, b)$. Functions satisfying this condition are said to be **continuous from the left**. If $f(x+) = f(x)$ at every $x \in (a, b)$, then f would be said to be **continuous from the right** (and this would be the case if the summation defined were to be taken over all indices n for which $x_n \leq x$!).

Figure 2: To Infinity and Beyond!

4.7   Infinite Limits and Limits at Infinity

Definition 4.54. For any $c \in \mathbb{R}$, the set of $x \in \mathbb{R}$ such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

Definition 4.55. Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$. We say

$$f(t) \to A \text{ as } t \to x,$$

where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that

1. $V \cap E$ is not empty, and

2. $f(t) \in U$ for all $t \in V \cap E$, where $t \neq x$.

> **Remark 4.56.** If $A, x \in \mathbb{R}$, this definition of a limit is the same as defined previously. As expected, properties of limits still hold in the extended real number system.

Theorem 4.57

Let f, g be defined on $E \subset \mathbb{R}$. Suppose $f(t) \to A$ and $g(t) \to B$ as $t \to x$. Then

1. $f(t) \to A'$ implies $A' = A$,

2. $(f + g)(t) \to A + B$,

3. $(fg)(t) \to AB$,

4. $(f/g)(t) \to A/B$,

provided that the right members of (2), (3), and (4) are defined. Note that $\infty - \infty, 0 \cdot \infty, \infty/\infty, A/0$ are not defined.

5   Differentiation

Note that in this chapter, we'll narrow our scope to real-valued functions defined on open and closed intervals.

5.1   The Derivative of a Real Function

Definition 5.1. Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$, define

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \quad (a < t < b, \ t \neq x),$$

provided this limit exists. We shall associate with f another function f' whose domain is the set of points x at which the limit exists; we'll call f' the **derivative** of f.

If f' is defined at a point x, we say that f is **differentiable** at x. If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E.

> **Remark 5.2.** Considering right and left hand limits of the quotient limit leads to the notion of right-hand and left-hand derivatives. At the endpoints a and b, the derivative, if it exists, is a right-hand or left-hand derivative, respectively.
>
> If f is defined on an open interval (a, b) and if $x \in (a, b)$, then $f'(x)$ is defined as expected; however, $f'(a)$ and $f'(b)$ are not defined.

> **Theorem 5.3**
>
> Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.

> **Remark 5.4.** Note that to prove equality of $a = b$, it is often easier to prove $a - b = 0$. We will do so here:
>
> As $t \to x$, we have that
>
> $$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \to f'(x) \cdot 0 = 0.$$

The converse of Theorem 5.3 is definitely not true (the quintessential example being $f(x) = |x|$). In fact, there exists a function that is continuous on all of $\mathbb{R}$ but is not differentiable at any point!

> **Theorem 5.5** (Sum, Product, and Quotient Rule)
>
> Suppose f,g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x, and
>
> 1. $(f + g)'(x) = f'(x) + g'(x)$;
>
> 2. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;
>
> 3. $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$
>
> of course assuming in (3) that $g(x) \neq 0$.

Remark 5.6. (1) is immediate. Notice that

$$f(t)g(t) - f(x)g(x) = f(t)[g(t) - g(x)] + g(x)[f(t) - f(x)],$$

and (2) follows. Notice that

$$\frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t - x} = \frac{1}{g(t)g(x)} \frac{f(t)g(x) - f(x)g(t)}{t - x} = \frac{1}{g(t)g(x)} \left[g(x)\frac{f(t) - f(x)}{t - x} - f(x)\frac{g(t) - g(x)}{t - x} \right],$$

and (3) follows.

Example 5.7

1. The derivative of any constant is 0.

2. If $f(x) = x$, then $f'(x) = 1$.

3. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$, for any $n \in \mathbb{Z}$. (This is called the **Power Rule**.)

4. Every polynomial is differentiable,

5. Every rational function is differentiable, except at points where the denominator is zero.

Theorem 5.8 (Chain Rule)

Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \quad (a \leq t \leq b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x).$$

Example 5.9

Define f as follows:

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

When $x \neq 0$, we get that $f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$. At $x = 0$, we must use the definition of a derivative to get that

$$\frac{f(t) - f(0)}{t - 0} = \sin\left(\frac{1}{t}\right),$$

which does not tend to any limit as $t \to 0$. So $f'(0)$ does not exist.

Figure 3: The cake is a lie.

> **Example 5.10**
>
> Define f as follows:
> $$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & (x \neq 0) \\ 0 & (x = 0), \end{cases}$$
> For $x \neq 0$, we get $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ and at $x = 0$, we get
> $$\left| \frac{f(t) - f(0)}{t - 0} \right| = \left| t \sin \frac{1}{t} \right| \leq |t|$$
> so as $t \to 0$, $f'(0) = 0$. Notice that f is differentiable for all x, but f' is not continuous (at $x = 0$).

5.2   Mean Value Theorems

Definition 5.11. Let f be a real function defined on a metric space X. We say that f has a **local maximum** at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$. (Local minima are defined similarly.)

> **Theorem 5.12**
>
> Let f be defined on $[a, b]$. If f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

Remark 5.13. To prove, choose $\delta > 0$ such that $f(q) \leq f(p)$ for all $p \in X$ with $d(p, q) < \delta$. Then
$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then
$$\frac{f(t) - f(x)}{t - x} \geq 0,$$
so as $t \to x$, $f'(x) \geq 0$. If $x < t < x + \delta$, then
$$\frac{f(t) - f(x)}{t - x} \leq 0,$$
so $f'(x) \leq 0$.

Theorem 5.14 (Generalized Mean Value Theorem)

If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there exists a point $x \in (a, b)$ where
$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Remark 5.15. Note that differentiability is *not* required at endpoints. To prove, let
$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \quad (a \leq t \leq b).$$
Notice that h is continuous on $[a, b]$, differentiable on (a, b), and
$$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$
To finish the proof, we need to show $h'(x) = 0$ for some $x \in (a, b)$. If h constant, we're done. If $h(t) > h(a)$ for some $t \in (a, b)$, then h attains its maximum at some x on the closed (compact) interval $[a, b]$. Since $h(a) = h(b)$, $x \in (a, b)$. If $h(t) < h(a)$, for some $t \in (a, b)$, the same argument applies to choose an x where h attains its minimum.

Theorem 5.16 (The Mean Value Theorem)

If f is a real continuous function on $[a, b]$, which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which
$$f(b) - f(a) = (b - a)f'(x).$$

Remark 5.17. The proof follows immediately after taking $g(x) = x$ in the Generalizeed Mean Value Theorem.

Theorem 5.18

Suppose f is differentiable in (a, b):

1. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.

2. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

3. If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

5.3   Continuity of Derivatives

Not every function is a derivative!

Derivatives which exist at every point in an interval have one important quality that they share with functions continuous on that interval: intermediate values are assumed.

> **Theorem 5.19**
>
> Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$. (A similar result holds if $f'(a) > f'(b)$.)

Remark 5.20. To prove, let
$$g(t) = f(t) - \lambda t.$$
Since $\lambda > f'(a)$, we know $g'(a) < 0$ so $g(t_1) < g(a)$ for some $t_1 \in (a, b)$. Similarly, $g(t_2) < g(b)$ for some $t_2 \in (a, b)$, so g attains minimum on (a, b). Thus there exists some $x \in (a, b)$ such that $g'(x) = 0$, and thus $0 = f'(x) - \lambda$.

> **Corollary 5.21**
>
> If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$. (But may very well have discontinuities of the second type. See the following example)

> **Example 5.22** (Deja vu)
>
> The function
> $$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$
> is differentiable but f' is discontinuous a $x = 0$.

5.4   L'Hospital's Rule

> **Theorem 5.23**
>
> Suppose f and g are real and differentiable in (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose
> $$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a.$$
> If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or if $g(x) \to +\infty$ as $x \to a$, then
> $$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$

Remark 5.24. Like most things in this chapter, you can use the Mean Value Theorem to prove this statement. That being said, I'm running on 2 hours of sleep and have been up for nearly 24 hours straight, so this one's not getting written down here.

5.5   Derivatives of Higher Order

Definition 5.25. If f has a derivative f' on an interval, and f' itself is differentiable, we denote the derivative of f' by f'' and call f'' the second derivative of f. We can continue in the same manner to obtain
$$f, f', f'', f(3), \ldots, f^{(n)},$$
where $f^{(n)}$ is called the nth derivative of f. In order for $f^{(n)}(x)$ to exist at a point x, $f^{(n-1)}$ must exist in a neighborhood of x, and be differentiable at x, and so on...

5.6   Taylor's Theorem

> **Theorem 5.26**
>
> Suppose f is a real function on $[a, b]$, $n \in \mathbb{N}$, $f^{(n-1)}$ is continuous on $[a, b]$, and $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$ and define
>
> $$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$
>
> Then there exists a point x between α and β such that
>
> $$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n.$$
>
> For $n = 1$, this is just the mean value theorem (but are we *that* surprised?)

In general, Taylor's Theorem lets us approximate functions with polynomials of degree $n - 1$, and (arguably most importantly,) estimate the error if we know bounds on $|f^{(n)}(x)|$.

Remark 5.27. Again, this uses the Mean Value Theorem in its proof. zzz

5.7   Differentiation of Vector-Valued Functions

Remark 5.28. In terms of vector valued functions, $\vec{f}(x)$ is the point of $\mathbb{R}^k$ (if there is one) for which

$$\lim_{t \to x} \left| \frac{\vec{f}(t) - \vec{f}(x)}{t - x} - \vec{f}'(x) \right| = 0,$$

and $\vec{f}'$ is again a function with values in $\mathbb{R}^k$. If $f_1, \ldots, f_k$ are the components of $\vec{f}$, then

$$\vec{f}' = (f_1', \ldots, f_k'),$$

and $\vec{f}$ is differentiable at a point x if, and only if, each of the functions $f_1, \ldots, f_k$ is differentiable at x.

Unfortunately, the Mean Value Theorem (and thus L'Hopital's Rule) don't necessarily hold in $\mathbb{R}^k$.

> **Example 5.29** (RIP the GOAT: MVT *was* the MVP)
>
> Let $f(x) = e^{ix} = \cos(x) + i\sin(x)$. Then
>
> $$f(2\pi) - f(0) = 1 - 1 = 0,$$
>
> but
>
> $$f'(x) = ie^{ix},$$
>
> (and thus $|f'(x)| = 1$ for all $x \in \mathbb{R}$.)

Example 5.30 (French Hospital est trés pauvre)

On the open interval, define $f(x) = x$ and

$$g(x) = x + x^2 e^{i/x^2}.$$

Notice that as $x \to 0$, both $f(x)$ and $g(x)$ approach 0. Since $|e^{it}| = 1$ for all $t \in \mathbb{R}$, notice that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1.$$

Next,

$$g'(x) = 1 + \left(2x - \frac{2i}{x}\right) e^{i/x^2} \quad (0 < x < 1),$$

so that

$$|g'(x)| \geq \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1.$$

Thus

$$\left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}$$

so

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$

Repose en paix, mon doux hôpital.

Theorem 5.31 (A New Hope)

Suppose $\vec{f}$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ and $\vec{f}$ is differentiable in (a, b). Then there exists $x \in (a, b)$ such that

$$|\vec{f}(b) - \vec{f}(a)| \leq (b - a)|\vec{f}'(x)|.$$

Remark 5.32. Let $\vec{z} = \vec{f}(b) - \vec{f}(a)$, and define

$$\phi(t) = \vec{z} \cdot \vec{f}(t) \quad (a \leq t \leq b).$$

Then ϕ is a real-valued continuous function on $[a, b]$ which is differentiable in (a, b). By the Mean Value Theorem (hooray! it's back!):

$$\phi(b) - \phi(a) = (b - a)\phi'(x) = (b - a)\vec{z} \cdot \vec{f}'(x)$$

for some $x \in (a, b)$. On the other hand,

$$\phi(b) - \phi(a) = \vec{z} \cdot \vec{f}(b) - z \cdot \vec{f}(a) = \vec{z} \cdot \vec{z} = |\vec{z}|^2.$$

Thus,

$$|\vec{z}|^2 = (b - a)|\vec{z} \cdot \vec{f}'(x)| \leq (b - a)|\vec{z}||\vec{f}'(x)|,$$

so $|\vec{z}| \leq (b - a)|\vec{f}'(x)|.$